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ON SUPER (*a*,*d*)-$\boldsymbol{C_3}$- ANTIMAGIC TOTAL LABELING OF DUTCH WINDMILL GRAPH $\boldsymbol{D_3^m}$

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ABSTRACT

This paper is aimed to investigate the existence of super $(a,d)-C_3-$ antimagic total labeling of dutch windmill graph D_3^m. The methods to achieves the goal was taken in three step. First of all determine the edge and vertices notation on dutch windmill graph . At the second step, labeling the vertices and edges of several dutch windmill graphs, then obtained the pattern. Finally pattern must be proven to become theorem. Based on the study, The Dutch Windmill Graph $D_3^m\,,with\ m \geq 2,$ has super $(14m+7,1)-C_3-$ antimagic total labeling, super $(13m+8,3)-C_3-$ antimagic total labeling, super $(12m+9,5)-C_3-$ antimagic total labeling, super $(11m+10,7)-C_3-$ antimagic total labeling, super $(10m+8,3)-C_3-$ antimagic total labeling.

Keywords: Dutch Windmill Graph, Circle Graph, Isomorphic Graph, Super $(a,d)-H-$ Antimagic Total Labeling

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PRELIMINARY

We denote $V(G)$ the set of vertices and $E(G)$ the set of edges of graph $G=(V(G),E(G))$. All graphs in this paper are finite, simple, and undirected. Let $|V(G)|=p$ and $|E(G)|=q$ be respectively the number of vertices and edges. The general reference for graph-theoretic can be seen in (D. B. West, 1996).

In graphs there are several subjects that are currently being investigated such as labeling and coloring. One of implementation of coloring graph in schedulling has been introduced (Cipta et al., 2023). Meanwhile, labeling can be implemented in solving problem in security system (Prihandoko et al., 2020). Labeling on a graph is a one-to-one mapping that assigns elements in graph into set of positive integers. Based on domain of labeling, we call labeling as a vertex labeling, or an edge labeling, or a total labeling(Baca & Miller, 2008) .Then the labeling developed including magic labeling and antimagic labeling. Magic labeling was first introduced by (Kotzig & Rosa, 1970). Meanwhile, antimagic labeling is the development of magic labeling carried out by (Hartsfield & Ringel Gerhard, 1990).

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Many of studies of magic and antimagic labeling has been investigated. Wheels are Cycle-Animagic was investigated by(Semaničová-Feňovčíková et al., 2015). $K-$ magicness of graphs has investigated by (Meenakshi & Kathiresan, 2019). Union of graphs with many three-path are also has antimagic labeling(Chavez et al., 2023). The magic valuation of the modified Watermill graph WM(n), is $\frac{1}{2}(21n+3)$ for n odd, n ≥ 3(Nurdin et al., 2018).

Let G and H be graphs. An $(a,d)-H-$antimagic total labeling of G is bijection function $\varphi: V(G) \cup E(G) \rightarrow \{1,2,\ldots, p+q\}$, such as for each subgraphs H' of isomorphic to H, the sum of label vertices and edges in H' forms arithmetic progression where a and d are some positif integer. This labeling is said to be super if the vertices are labeled by smallest labels. In this case we say G is *super* $(a,d)-H-$*antimagic* (Inayah et al., 2009).

A number of classification studies on super $(a,d)-H$ antimagic has been extensively investigated(Rajkumar et al., 2022). Subdivision of a Fan Graph has super $(a,d)-$ Cycle $-$ antimagic labelling (Prihandini et al., 2018). A super (a,d)-Bm-antimagic total covering of a generalized amalgamation of Fan Graphs was discovered by (Agustin et al., 2017) An $(a,d)-H-$antimagic total labeling on Double Cones was investigated(Roswitha et al., 2019). Super $(a,d)\ -H-$ antimagic labeling of subdivided graph was discovered by(Taimur et al., 2018). Super sub division of cycle also has super $(a,d)-\ H-$ antimagic total labeling(Bhatavadekar, 2021). Path Chain Graph has super – *(a,d)* –*H* $-$ antimagic total labeling(Zhu & Liang, 2021). disconnected graph mCn is super

$(a,d) - Cn$ –antimagic total labeling(Susanto, 2018). Super antimagic total labeling for comb product of graphs also studied (Agustin et al., 2019) . A In the continuation of this study, this paper will investigate super (a, d) − C_3 − antimagic total labeling of the Dutch Windmill Graph D_3^m.

The Dutch Windmill graph is a graph obtained by taking *m* copies of the circle graph C_n with one overlapping vertex, and denoted by D_n^m (Mohammed et al., 2020). Figure 1 is a picture of a Dutch Windmill D_3^3.

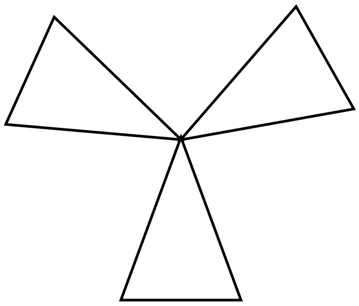

Figure 1. The Dutch Windmill Graph D_3^3

METHODS

In this section we provide the definition of an $(a,d) - H$ –antimagic total labeling as basis we use in this research . An edge covering of G is a family of different subgraphs $H_1, H_2, H_3, \dots, H_m$ such that any edege of $E(G)$ belongs to at least one of the subgraphs $H_k, 1 \le k \le m$. If H_k are isomorphic to a given graph H, then G admits an H –covering.

An $(a,d) - H$ – antimagic total labeling G is a bijective function $\varphi: V(G) \cup E(G) \rightarrow \{1,2,\dots, |V(G)| + |E(G)|\}$ such that for each subgraphs H' that are isomorphic to H, the sum of label vertices and edges in H' is

$$wt(H') = \sum_{v \in V(H')} \varphi(v) + \sum_{e \in E(H')} \varphi(e) \quad (1)$$

form arithmetic progression, $\{a, a + d, a + 2d, \dots, a + (n-1)d\}$, where a and d are positive integers and n is the number of subgraphs of G that is isomorphic to H. This labeling is said to be super if the vertices on graph G is labeled with the smallest labels.

In this study, authors searched some literatures related to total labeling then we starting labeled Dutch Windmill graph by setting the value of difference. From several example of Dutch Windmill graph that have been labeled, then pattern will be obtained.

This pattern is expressed to be conjecture and must be proved to be true. If it is true, then conjecture become theorem. If the truth value of conjecture is false, then author must construct other conjecture until the correct conjecture obtained. The flowchart of this research is described below.

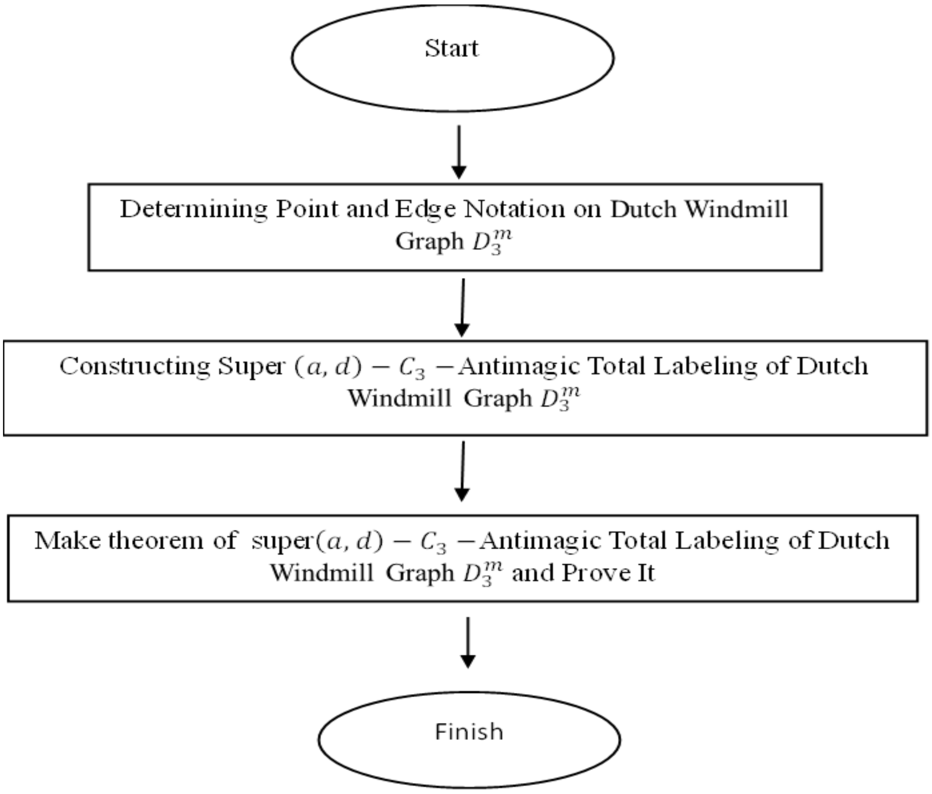

Figure 2. Flowchart

RESULT AND DISCUSSION

The Dutch Windmill graph D_3^m has set of vertices denoted by $V(D_3^m) = \{c\} \cup \{v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\}$ and set of edges denoted by $E(D_3^m) = \{cv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{v_1^j v_2^j | 1 \leq j \leq m\}$. Figure 3 is a picture of vertices and edges notation of Dutch Windmill Graph D_3^m.

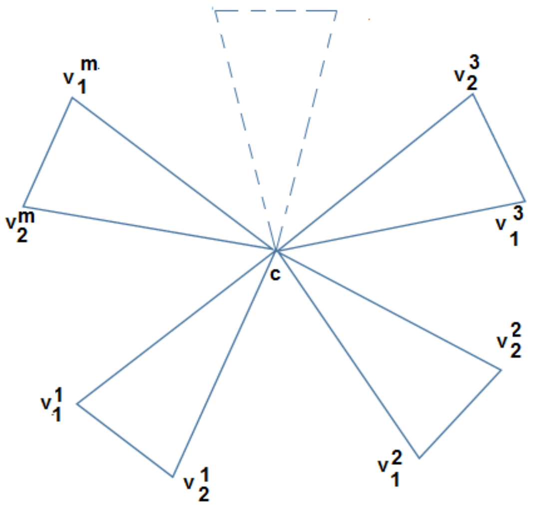

Figure 3. Vertices and Edges Notation of Dutch Windmill Graph D_3^m

Theorem 1 The Dutch Windmill Graph D_3^m has super $(14m+7,1)-C_3-$ antimagic total labeling for $m \geq 2$.

Proof :

Defined total labeling on $f_1 : V(D_3^m) \cup E(D_3^m) \to \{1,2,3,\dots,|V(D_3^m)|+|E(D_3^m)|\}$ as follows:

1. Labeling of set of vertices $V(D_3^m) = \{c\} \cup \{v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\}$ is defined as follows

   $f_1(c) = 2m+1$

   $$f_1(v_i^j) = \begin{cases} 2j-1, & 1 \leq j \leq m, \quad i=1 \\ 2j, & 1 \leq j \leq m, \quad i=2 \end{cases}$$

2. Labeling of set of edges $E(D_3^m) = \{cv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{v_1^j v_2^j | 1 \leq j \leq m\}$ is defined as follows

   $f_1(v_1^j v_2^j) = 4m+1+j, 1 \leq j \leq m$

   $$f_1(cv_i^j) = \begin{cases} 4m+3-2j, & 1 \leq j \leq m, \quad i=1 \\ 4m+2-2j, & 1 \leq j \leq m, \quad i=2 \end{cases}$$

It can be seen that $f_1(V(D_3^m)) = [1, |V(D_3^m)|]$. Defined $C_3^{(j)}$ for $1 \leq j \leq m$ is subgraph of graph D_3^m isomorphic to graph C_3 with its set of vertices $V(C_3^{(j)}) = \{c, v_1^j, v_2^j\}$ and its set of edges $E(C_3^{(j)}) = \{cv_1^j, cv_2^j, v_1^j v_2^j\}$.

Then, the weight of subgraph $C_3^{(j)}$ is obtained by the formula $wt(C_3^{(j)}) = f_1(c) + f_1(v_1^j) + f_1(v_2^j) + f_1(cv_1^j) + f_1(cv_2^j) + f_1(v_1^j v_2^j)$, for $1 \leq j \leq m$. So

obtained,

$wt\left(C_3^{(j)}\right) = (2m+1) + (2j-1) + (2j) + (4m+3-2j) + (4m+2-2j) + (4m+1+j) = 14m + j + 6$, untuk $1 \le j \le m$

.

The value of difference d from super $(a,d) - C_3 -$ antimagic total labeling of the dutch windmill graph D_3^m is obtained by the formula $d = wt\left(C_3^{(j+1)}\right) - wt\left(C_3^{(j)}\right)$ for $j = 1,2,..,m-1$, and the value of a is obtained by the formula $a = wt\left(C_3^{(1)}\right)$. So, $d = (14m + j + 1 + 6) - (14m + j + 6) = 1$ and $a = wt\left(C_3^{(1)}\right) = 14m + 1 + 6 = 14m + 7$. Thus, it is proven that the Dutch Windmill graph D_3^m has a Super $(14m + 7, 1) - C_3$ – antimagic total labeling for $m \ge 2$. ■

Illustration of super $(77, 1) - C_3 -$ antimagic total labeling of dutch windmill graph D_3^5 in Figure 4.

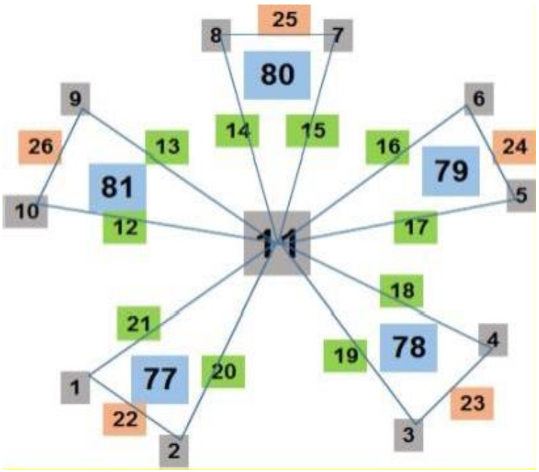

Figure 4. Super (77, 1) $- C_3 -$ Antimagic Total Labeling of Dutch Windmill Graph D_3^5

Based on vertices and edges labeling defined in proof above, then vertices labeling for Dutch Windmill Graph D_3^5 in figure 4, respectively are $f_1(c) = 11$, $f_1(v_1^1) = 1, f_1(v_2^1) = 2$, $f_1(v_1^2) = 3, f_1(v_2^2) = 4$, $f_1(v_1^3) = 5, f_1(v_2^3) = 6$, $f_1(v_1^4) = 7, f_1(v_2^4) = 8$, $f_1(v_1^5) = 9, f_1(v_2^5) = 10$, shown di grey color. Edges labeling shown in orange color, respectively are $f_1(v_1^1v_2^1) = 22$, $f_1(v_1^2v_2^2) = 23$, $f_1(v_1^3v_2^3) = 24$, $f_1(v_1^4v_2^4) = 25$, $f_1(v_1^5v_2^5) = 26$. Edges labeling shown in green color, respectively are $f_1(cv_1^1) = 21$, $f_1(cv_2^1) = 20$, $f_1(cv_1^2) = 19$, $f_1(cv_2^2) = 18$, $f_1(cv_1^3) = 17, f_1(cv_2^3) = 16, f_1(cv_1^4) = 15, f_1(cv_2^4) = 14$, $f_1(cv_1^5) = 13$, $f_1(cv_2^5) = 12$. The weight of subgraph $C_3^{(j)}$ is obtained by the formula $wt(C_3^j) = f_1(c) + f_1(v_1^j) + f_1(v_2^j) + f_1(cv_1^j) + f_1(cv_2^j) + f_1(v_1^jv_2^j)$, then

$wt(C_3^1) = 11 + 1 + 2 + 22 + 21 + 20 = 77,$

$wt(C_3^2) = 11 + 3 + 4 + 23 + 19 + 18 = 78,$

$wt(C_3^3) = 11 + 5 + 6 + 24 + 17 + 16 = 79,$

$wt(C_3^4) = 11 + 7 + 8 + 25 + 15 + 14 = 80,$

$wt(C_3^5) = 11 + 9 + 10 + 26 + 13 + 12 = 81,$ shown in blue color. The weight of subgraph form arithmetic progression, $77, 78, 79, 80, 81$ with value of $a = 77$, and $d = 1$. It shown that Dutch Windmill Graph D_3^m, with $m = 5$ has Super $(14m + 7, 1) - C_3 -$ Antimagic Total Labeling.

Theorem 2 The Dutch Windmill Graph D_3^m has super $(13m + 8,3) - C_3 -$ antimagic total labeling for $m \geq 2$.

Proof :

Defined total labeling on $f_1 : V(D_3^m) \cup E(D_3^m) \to \{1, 2, 3, \dots, |V(D_3^m)| + |E(D_3^m)|\}$ as follows:

1. Labeling of set of vertices $V(D_3^m) = \{c\} \cup \{v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\}$ is defined as follows

$$f_2(c) = 2m + 1$$

$$f_2(v_i^j) = \begin{cases} m + 1 - j, & 1 \leq j \leq m, \quad i = 1 \\ m + j, & 1 \leq j \leq m, \quad i = 2 \end{cases}$$

2. Labeling of set of edges $E(D_3^m) = \{cv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{v_1^j v_2^j | 1 \leq j \leq m\}$ is defined as follows

$$f_2(v_1^j v_2^j) = 3m + 2 - j, 1 \leq j \leq m$$

$$f_2(cv_i^j) = \begin{cases} 3m + 1 + 2j, & 1 \leq j \leq m, \quad i = 1 \\ 3m + 2j, & 1 \leq j \leq m, \quad i = 2 \end{cases}$$

It can be seen that $f_2(V(D_3^m)) = [1, |V(D_3^m)|]$. Defined $C_3^{(j)}$ for $1 \leq j \leq m$ is subgraph of graph D_3^m isomorphic to graph C_3 with its set of vertices $V(C_3^{(j)}) = \{c, v_1^j, v_2^j\}$ and its set of edges $E(C_3^{(j)}) = \{cv_1^j, cv_2^j, v_1^j v_2^j\}$.

Then, the weight of subgraph $C_3^{(j)}$ is obtained by the formula $wt(C_3^{(j)}) = f_2(c) + f_2(v_1^j) + f_2(v_2^j) + f_2(cv_1^j) + f_2(cv_2^j) + f_2(v_1^j v_2^j)$, for $1 \leq j \leq m$. So obtained,

$wt(C_3^{(j)}) = (2m + 1) + (m + 1 - j) + (m + j) + (3m + 1 + 2j) + (3m + 2j) + (3m + 2 - j) = 13m + 3j + 5,$ for $1 \leq j \leq m$.

The value of difference d from super $(a, d) - C_3 -$ antimagic total labeling of the D_3^m is obtained by the formula $d = wt\left(C_3^{(j+1)}\right) - wt\left(C_3^{(j)}\right)$ for $j = 1,2,..,m-1,$ and the value of a is obtained by the formula $a = wt\left(C_3^{(1)}\right)$. So, $d = (13m + 3j + 3 + 5) - (13m + 3j + 5) = 3$ and $a = wt\left(C_3^{(1)}\right) = 14m + 3 + 5 = 13m + 8$. Thus, it is proven that the Dutch Windmill Graph D_3^m has a super $(13m + 8, 3) - C_3$ –antimagic total labeling for $m \geq 2$. ■

Illustration of Super $(73, 3) - C_3 -$ Antimagic Total Labeling Dutch Windmill Graph D_3^5 in Figure 5.

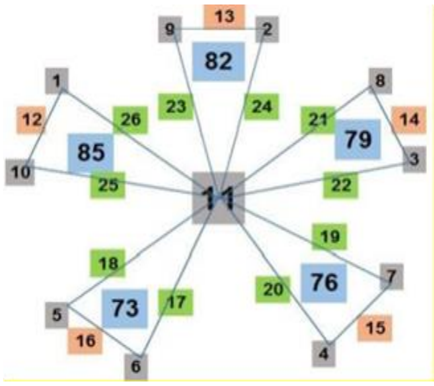

Figure 5. Super (73, 3) − $C3$ − Antimagic Total Labeling of Dutch Windmill Graph D_3^5

Based on vertices and edges labeling defined in proof above, then vertices labeling for Dutch Windmill Graph D_3^5 in figure 5, respectively are $f_2(c) = 11$, $f_2(v_1^1) = 5, f_2(v_2^1) = 6$, $f_2(v_1^2) = 4, f_2(v_2^2) = 7$, $f_2(v_1^3) = 3, f_2(v_2^3) = 8$, $f_2(v_1^4) = 2, f_2(v_2^4) = 9$, $f_2(v_1^5) = 1, f_2(v_2^5) = 10$, shown di grey color. Edges labeling shown in orange color, respectively are $f_2(v_1^1 v_2^1) = 16$, $f_2(v_1^2 v_2^2) = 15$, $f_2(v_1^3 v_2^3) = 14$, $f_2(v_1^4 v_2^4) = 13$, $f_2(v_1^5 v_2^5) = 12$. Edges labeling shown in green color, respectively are $f_2(cv_1^1) = 18$, $f_2(cv_2^1) = 17$, $f_2(cv_1^2) = 20$, $f_2(cv_2^2) = 19$, $f_2(cv_1^3) = 22, f_2(cv_2^3) = 21, f_2(cv_1^4) = 24, f_2(cv_2^4) = 23$, $f_2(cv_1^5) = 26$, $f_2(cv_2^5) = 25$. The weight of subgraph C_3^j is obtained by the formula $wt(C_3^j) = f_2(c) + f_2(v_1^j) + f_2(v_2^j) + f_2(cv_1^j) + f_2(cv_2^j) + f_2(v_1^j v_2^j)$, then

$wt(C_3^1) = 11 + 5 + 6 + 16 + 18 + 17 = 73$,

$wt(C_3^2) = 11 + 4 + 7 + 15 + 20 + 19 = 76$,

$wt(C_3^3) = 11 + 3 + 8 + 14 + 22 + 21 = 79$,

$wt(C_3^4) = 11 + 2 + 9 + 13 + 24 + 23 = 82$,

$wt(C_3^5) = 11 + 1 + 10 + 12 + 25 + 26 = 85$, shown in blue color. The weight of subgraph form arithmetic progression, $73, 76, 79, 82, 85$ with value of $a = 73$, and $d = 3$. It shown that Dutch Windmill Graph D_3^m, with $m = 5$ has Super $(13m + 8,3) - C_3 -$ Antimagic Total Labeling.

Theorem 3 The Dutch Windmill Graph D_3^m has super $(12m + 9) - C_3 -$ antimagic total labeling for $m \geq 2$.

Proof :

Defined total labeling on $f_3 : V(D_3^m) \cup E(D_3^m) \to \{1, 2, 3, \dots, |V(D_3^m)| + |E(D_3^m)|\}$ as follows:

1. Labeling of set of vertices $V(D_3^m) = \{c\} \cup \{v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\}$ is defined as follows

$$f_3(c) = 2m + 1$$

$$f_3(v_i^j) = \begin{cases} 2m + 1 - 2j, & 1 \leq j \leq m, \quad i = 1 \\ 2m + 2 - 2j, & 1 \leq j \leq m, \quad i = 2 \end{cases}$$

2. Labeling of set of edges $E(D_3^m) = \{cv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{v_1^j v_2^j | 1 \leq j \leq m\}$ is defined as follows

$$f_3(v_1^j v_2^j) = 2m + 3j, 1 \leq j \leq m$$

$$f_3(cv_i^j) = \begin{cases} 2m + 1 + 3j, & 1 \leq j \leq m, \quad i = 1 \\ 2m - 1 + 3j, & 1 \leq j \leq m, \quad i = 2 \end{cases}$$

It can be seen that $f_3(V(D_3^m)) = [1, |V(D_3^m)|]$. Defined $C_3^{(j)}$ for $1 \leq j \leq m$ is subgraph of graph D_3^m isomorphic to graph C_3 with its set of vertices $V\left(C_3^{(j)}\right) = \{c, v_1^j, v_2^j\}$ and its set of edges $E\left(C_3^{(j)}\right) = \{cv_1^j, cv_2^j, v_1^j v_2^j\}$.

Then, the weight of subgraph $C_3^{(j)}$ is obtained by the formula

$wt\left(C_3^{(j)}\right) = f_3(c) + f_3(v_1^j) + f_3(v_2^j) + f_3(cv_1^j) + f_3(cv_2^j) + f_3(v_1^j v_2^j)$, for $1 \leq j \leq m$, so obtained

$$wt\left(C_3^{(j)}\right) = (2m + 1) + (2m + 1 - 2j) + (2m + 2 - 2j) + (2m + 1 + 3j) + (2m - 1 + 3j) + (2m + 3j) = 12m + 5j + 4, \text{for } 1 \leq j \leq m.$$

The value of difference d from Super $(a, d) - C_3 -$ antimagic total labeling of D_3^m is obtained by the formula $d = wt\left(C_3^{(j+1)}\right) - wt\left(C_3^{(j)}\right)$ for $j = 1,2, \dots, m - 1$, and the value of a is obtained by the formula $a = wt\left(C_3^{(1)}\right)$. So, $d = (12m + 5j + 5 + 4) - (12m + 5j + 4) = 5$ and

$a = wt\left(C_3^{(1)}\right) = 12m + 4 + 5 = 12m + 9$. Thus, it is proven that the dutch windmill Graph D_3^m has a super $(12m + 9, 5) - C_3$ –antimagic total labeling for $m \geq 2$. ■

Illustration of Super $(69,5) - C_3 -$ Antimagic Total Labelling of Dutch Windmill Graph D_3^5 in Figure 6.

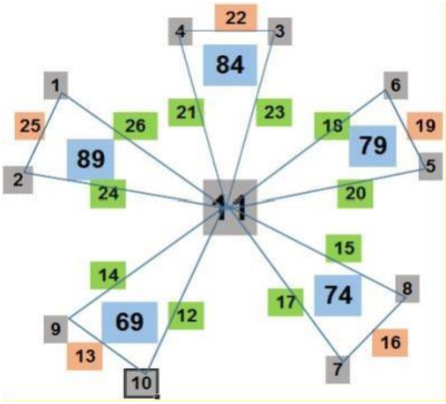

Figure 6. Super (69,5) − $C3$ – Antimagic Total Labeling of Dutch Windmill Graph $\boldsymbol{D_3^5}$

Based on vertices and edges labeling defined in proof above, then vertices labeling for Dutch Windmill Graph D_3^5 in figure 6, respectively are $f_3(c) = 11$, $f_3(v_1^1) = 9, f_3(v_2^1) = 10$, $f_3(v_1^2) = 7, f_3(v_2^2) = 8$, $f_3(v_1^3) = 5, f_3(v_2^3) = 6$, $f_3(v_1^4) = 3, f_3(v_2^4) = 4$, $f_3(v_1^5) = 1, f_2(v_2^5) = 2$, shown di grey color. Edges labeling shown in orange color, respectively are $f_3(v_1^1 v_2^1) = 13$, $f_3(v_1^2 v_2^2) = 16$, $f_3(v_1^3 v_2^3) = 19$, $f_3(v_1^4 v_2^4) = 22$, $f_3(v_1^5 v_2^5) = 25$. Edges labeling shown in green color, respectively are $f_3(cv_1^1) = 14$, $f_3(cv_2^1) = 12$, $f_3(cv_1^2) = 17$, $f_3(cv_2^2) = 15$, $f_3(cv_1^3) = 20, f_3(cv_2^3) = 18, f_3(cv_1^4) = 23, f_3(cv_2^4) = 21$, $f_3(cv_1^5) = 26$, $f_3(cv_2^5) = 24$. The weight of subgraph C_3^j is obtained by the formula $wt(C_3^j) = f_3(c) + f_3(v_1^j) + f_3(v_2^j) + f_3(cv_1^j) + f_3(cv_2^j) + f_3(v_1^j v_2^j)$, then

$wt(C_3^1) = 11 + 9 + 10 + 14 + 12 + 13 = 69$,

$wt(C_3^2) = 11 + 7 + 8 + 17 + 15 + 16 = 74$,

$wt(C_3^3) = 11 + 5 + 6 + 20 + 18 + 19 = 79$,

$wt(C_3^4) = 11 + 3 + 4 + 23 + 21 + 22 = 84$,

$wt(C_3^5) = 11 + 1 + 2 + 26 + 24 + 25 = 89$, shown in blue color. The weight of subgraph form arithmetic progression, $69, 74, 79, 84, 89$ with value of $a = 69$, and $d = 5$. It shown that Dutch Windmill Graph D_3^m, with $m = 5$ has Super $(12m + 9{,}5) - C_3 -$ Antimagic Total Labeling.

Theorem 4 The Dutch Windmill Graph D_3^m has super $(11m + 10{,}7) - C_3 -$ antimagic total labeling for $m \geq 2$.

Proof

Defined total labeling on $f_3 : V(D_3^m) \cup E(D_3^m) \rightarrow \{1, 2, 3, \dots, |V(D_3^m)| + |E(D_3^m)|\}$ as follows:

1. Labeling of set of vertices $V(D_3^m) = \{c\} \cup \{v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\}$ is defined as follows

$$f_4(c) = 2m + 1$$

$$f_4(v_i^j) = \begin{cases} 2j - 1, & 1 \leq j \leq m, \ i = 1 \\ 2j, & 1 \leq j \leq m, \ i = 2 \end{cases}$$

2. Labeling of set of edges $E(D_3^m) = \{cv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{v_1^j v_2^j | 1 \leq j \leq m\}$ is defined as follows

$$f_4(v_1^j v_2^j) = 3m + 2 - j, 1 \leq j \leq m$$

$$f_4(cv_i^j) = \begin{cases} 3m + 2j, & 1 \leq j \leq m, \ i = 1 \\ 3m + 2j + 1, & 1 \leq j \leq m, \ i = 2 \end{cases}$$

It can be seen that $f_4(V(D_3^m)) = [1, |V(D_3^m)|]$. Defined $C_3^{(j)}$ for $1 \leq j \leq m$ is subgraph of graph D_3^m isomorphic to graph C_3 with its set of vertices $V\left(C_3^{(j)}\right) = \{c, v_1^j, v_2^j\}$ and its set of edges $E\left(C_3^{(j)}\right) = \{cv_1^j, cv_2^j, v_1^j v_2^j\}$.

Then, the weight of subgraph $C_3^{(j)}$ is obtained by the formula $wt\left(C_3^{(j)}\right) = f_4(c) + f_4(v_1^j) + f_4(v_2^j) + f_4(cv_1^j) + f_4(cv_2^j) + f_4(v_1^j v_2^j)$, for $1 \leq j \leq m$, so obtained

$$wt\left(C_3^{(j)}\right) = (2m + 1) + (2j - 1) + (2j) + (3m + 2j) + (3m + 1 + 2j) + (3m + 2 - j)$$

$$= 11m + 7j + 3, \text{for } 1 \leq j \leq m.$$

The value of difference d from super $(a, d) - C_3 -$ antimagic total labeling of D_3^m is obtained by the formula $d = wt\left(C_3^{(j+1)}\right) - wt\left(C_3^{(j)}\right)$ for $j = 1, 2, \dots, m - 1$, and the value of a is obtained by the formula $a = wt\left(C_3^{(1)}\right)$. So, $d = (11m + 7j + 7 + 3) - (11m + 7j + 3) = 7$ and $a = wt\left(C_3^{(1)}\right) = 11m + 7 + 3 = 11m + 10$. Thus, it is proven that the Dutch Windmill Graph D_3^m has a super $(11m + 10, 7) - C_3$ –antimagic total labeling for $m \geq 2$. ■

Illustration of Total Labeling $(65{,}7) - C_3 -$ Antimagic Super Dutch Windmill Graph D_3^5 in Figure 7.

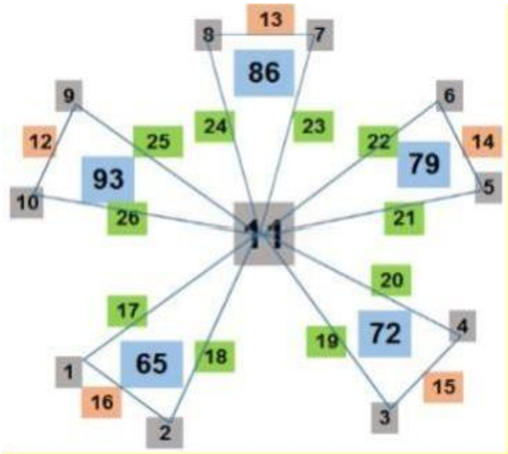

Figure 7. Super (65,7) − $C3$ − Antimagic Total Labeling of Dutch Windmill Graph D_3^5

Based on vertices and edges labeling defined in proof above, then vertices labeling for Dutch Windmill Graph D_3^5 in figure 7, respectively are $f_4(c) = 11$, $f_4(v_1^1) = 1, f_4(v_2^1) = 2$, $f_4(v_1^2) = 3, f_4(v_2^2) = 4$, $f_4(v_1^3) = 5, f_4(v_2^3) = 6$, $f_4(v_1^4) = 7, f_4(v_2^4) = 8$, $f_4(v_1^5) = 9, f_4(v_2^5) = 10$, shown di grey color. Edges labeling shown in orange color, respectively are $f_4(v_1^1 v_2^1) = 16$, $f_4(v_1^2 v_2^2) = 15$, $f_4(v_1^3 v_2^3) = 14$, $f_4(v_1^4 v_2^4) = 13$, $f_4(v_1^5 v_2^5) = 12$. Edges labeling shown in green color, respectively are $f_4(cv_1^1) = 17$, $f_4(cv_2^1) = 18$, $f_4(cv_1^2) = 19$, $f_4(cv_2^2) = 20$, $f_4(cv_1^3) = 21, f_4(cv_2^3) = 22, f_4(cv_1^4) = 23, f_4(cv_2^4) = 24$, $f_4(cv_1^5) = 25$, $f_4(cv_2^5) = 26$. The weight of subgraph C_3^j is obtained by the formula $wt(C_3^j) = f_4(c) + f_4(v_1^j) + f_4(v_2^j) + f_4(cv_1^j) + f_4(cv_2^j) + f_4(v_1^j v_2^j)$, then

$wt(C_3^1) = 11 + 1 + 2 + 17 + 18 + 16 = 65$,

$wt(C_3^2) = 11 + 3 + 4 + 19 + 20 + 15 = 72$,

$wt(C_3^3) = 11 + 5 + 6 + 21 + 22 + 14 = 79$,

$wt(C_3^4) = 11 + 7 + 8 + 23 + 24 + 13 = 86$,

$wt(C_3^5) = 11 + 9 + 10 + 25 + 26 + 12 = 93$, shown in blue color. The weight of subgraph form arithmetic progression, $65, 72, 79, 86, 93$ with value of $a = 65$, and $d = 7$. It shown that Dutch Windmill Graph D_3^m, with $m = 5$ has Super $(11m + 10, 7) - C_3 -$ Antimagic Total Labeling.

Theorem 5 The Dutch Windmill Graph D_3^m has super $(10m + 11,9) - C_3 -$ antimagic total labelling for every integer m ≥ 2.

Proof

Defined total labeling on $f_3 : V(D_3^m) \cup E(D_3^m) \to \{1,2,3,\ldots,|V(D_3^m)| + |E(D_3^m)|\}$ as follows:

1. Labeling of set of vertices $V(D_3^m) = \{c\} \cup \{v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\}$ is defined as follows

   $f_5(c) = 2m + 1$

$$f_5(v_i^j) = \begin{cases} 2j - 1, & 1 \leq j \leq m, \quad i = 1 \\ 2j, & 1 \leq j \leq m, \quad i = 2 \end{cases}$$

2. Labeling of set of edges $E(D_3^m) = \{cv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{v_1^j v_2^j | 1 \leq j \leq m\}$ is defined as follows

$$f_5(v_1^j v_2^j) = 2m + 1 + j, 1 \leq j \leq m$$

$$f_5(cv_i^j) = \begin{cases} 3m + 2j, & 1 \leq j \leq m, \quad i = 1 \\ 3m + 2j + 1, & 1 \leq j \leq m, \quad i = 2 \end{cases}$$

It can be seen that $f_5(V(D_3^m)) = [1, |V(D_3^m)|]$. Defined $C_3^{(j)}$ for $1 \leq j \leq m$ is subgraph of graph D_3^m isomorphic to graph C_3 with its set of vertices $V\left(C_3^{(j)}\right) = \{c, v_1^j, v_2^j\}$ and its set of edges $E\left(C_3^{(j)}\right) = \{cv_1^j, cv_2^j, v_1^j v_2^j\}$. Then, the weight of subgraph $C_3^{(j)}$ is obtained by the formula $wt\left(C_3^{(j)}\right) = f_5(c) + f_5(v_1^j) + f_5(v_2^j) + f_5(cv_1^j) + f_5(cv_2^j) + f_5(v_1^j v_2^j)$, for $1 \leq j \leq m$, so obtained

$$wt\left(C_3^{(j)}\right) = (2m + 1) + (2j - 1) + (2j) + (3m + 2j) + (3m + 1 + 2j) + (2m + 1 + j)$$

$$= 10m + 9j + 2, \quad \text{for } 1 \leq j \leq m.$$

The value of difference d from super $(a, d) - C_3 -$ antimagic total labeling on graph D_3^m is obtained by the formula $d = wt\left(C_3^{(j+1)}\right) - wt\left(C_3^{(j)}\right)$ for $j = 1,2,\ldots,m - 1$, and the value of a is obtained by the formula $a = wt\left(C_3^{(1)}\right)$. So, $d = (10m + 9j + 9 + 2) - (10m + 9j + 2) = 9$ and $a = wt\left(C_3^{(1)}\right) = 10m + 9 + 2 = 10m + 11$. Thus, it is proven that the Dutch Windmill Graph D_3^m has a super $(11m + 10, 7) - C_3$ –antimagic total labeling for $m \geq 2$. ■

Illustration of Total Labeling $(61,9) - C_3 -$ Antimagic Super Dutch Windmill Graph D_3^5 in Figure 8.

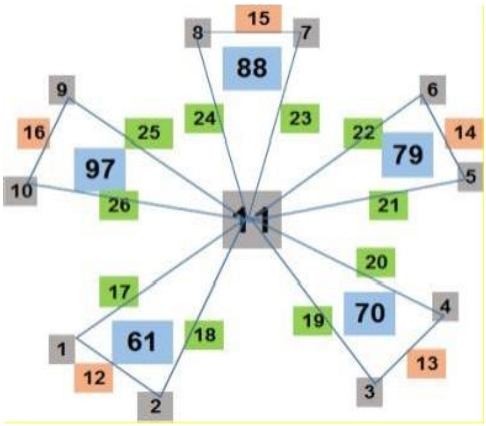

Figure 8. Super (61,9) − C_3− Antimagic Total Labelling of Dutch Windmill Graph D_3^5

Based on vertices and edges labeling defined in proof above, then vertices labeling for Dutch Windmill Graph D_3^5 in figure 7, respectively are $f_5(c) = 11$, $f_5(v_1^1) = 1, f_5(v_2^1) = 2$, $f_5(v_1^2) = 3, f_5(v_2^2) = 4$, $f_5(v_1^3) = 5, f_5(v_2^3) = 6$, $f_5(v_1^4) = 7, f_5(v_2^4) = 8$, $f_5(v_1^5) = 9, f_5(v_2^5) = 10$, shown di grey color. Edges labeling shown in orange color, respectively are $f_5(v_1^1 v_2^1) = 12$, $f_5(v_1^2 v_2^2) = 13$, $f_5(v_1^3 v_2^3) = 14$, $f_5(v_1^4 v_2^4) = 15$, $f_5(v_1^5 v_2^5) = 16$. Edges labeling shown in green color, respectively are $f_5(cv_1^1) = 17$, $f_5(cv_2^1) = 18$, $f_5(cv_1^2) = 19$, $f_5(cv_2^2) = 20$, $f_5(cv_1^3) = 21, f_5(cv_2^3) = 22, f_5(cv_1^4) = 23, f_5(cv_2^4) = 24$, $f_5(cv_1^5) = 25$, $f_5(cv_2^5) = 26$. The weight of subgraph C_3^j is obtained by the formula $wt(C_3^j) = f_5(c) + f_5(v_1^j) + f_5(v_2^j) + f_5(cv_1^j) + f_5(cv_2^j) + f_5(v_1^j v_2^j)$, then

$wt(C_3^1) = 11 + 1 + 2 + 17 + 18 + 12 = 61$,

$wt(C_3^2) = 11 + 3 + 4 + 19 + 20 + 13 = 70$,

$wt(C_3^3) = 11 + 5 + 6 + 21 + 22 + 14 = 79$,

$wt(C_3^4) = 11 + 7 + 8 + 23 + 24 + 15 = 88$,

$wt(C_3^5) = 11 + 9 + 10 + 25 + 26 + 16 = 97$, shown in blue color. The weight of subgraph form arithmetic progression, $61, 70, 79, 88, 97$ with value of $a = 61$, and $d = 9$. It shown that Dutch Windmill Graph D_3^m, with $m = 9$ has Super $(10m + 11, 7) - C_3 -$ Antimagic Total Labeling.

CONCLUSION

Based on the previous discussion, it can be concluded The Dutch Windmill Graph D_3^m with $m \geq 2$ has super $(a, d) - C_3$−antimagic total labeling for the values $d = 1, 3, 5, 7, and\ 9$. The values of a and d from the results of total labeling $(a, d) - C3$

−super magic of the Dutch Windmill Graph D_3^m with $m \geq 2$ is presented in the table below.

Table 1. Value of *a* and *d*

d	***a***
1	$14m + 7$
3	$13m + 8$
5	$12m + 9$
7	$11m + 10$
9	$10m + 11$

The value of a for $d > 9$ have not been obtained, so we suggest to investigate super $(a, d) - C_3$ −antimagic total labeling of Dutch Windmill Graph, with $m \geq 2$, for $d > 9$, and d odd and also we suggest to investigate super $(a, d) - C_3$ −antimagic total labeling of the Dutch Windmill Graph D_3^m, with $m \geq 2$, for d $even$.

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