M A T H L I N E
JURNAL MATEMATIKA DAN PENDIDIKAN MATEMATIKA

ISSN 2502-5872 (Print)
ISSN 2622-3627 (Elektronik)

Volume 11 Number 2, May 2026, 307-316

NUMERICAL SOLUTION ANALYSIS USING THE RUNGE-KUTTA-FEHLBERG METHOD ON A MODEL OF THE SPREAD OF SMOKING BEHAVIOR

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ABSTRACT

We consider a mathematical model of the spread of smoking behavior in the population. The model is solved using the Runge-Kutta-Fehlberg numerical method. As smoking is not healthy in general, we are interested in the case of smoker-free in the long term of time. The equilibrium of the smoker-free is recalled and used to match the numerical simulation for large time values. In addition, our contribution in this paper is to verify that the Runge-Kutta-Fehlberg numerical method is fifth order in convergence. Using error tabulation, we show that the numerical method is indeed of the fifth order.

Keywords: Accuracy, Epidemics, Equilibrium, Infectious, Susceptible

How to Cite: Lestari, Y. D & Mungkasi, S. (2026). Numerical Solution Analysis Using The Runge-Kutta-Fehlberg Method on A Model of The Spread of Smoking Behavior. *Mathline: Jurnal Matematika dan Pendidikan Matematika, 11*(2), 307-316. http://doi.org/10.31943/mathline.v11i2.1050

PRELIMINARY

Smoking behavior exists in the society (BPS, 2023; World of Statistics, 2023). Some people do smoking for some reasons, such as to overcome anxiety or just a habit in their daily life. Generally, smokes contain nicotine (Alegantina, 2018) and other chemical substances. Despite the negative effects of smoking, some people keep smoking.

Researchers studied that smoking behavior is contagious. Ullah et al. (2016) developed a mathematical model for the spread of smoking behavior into a system of nonlinear ordinary differential equations. Günerhan et al. (2022) modified the work of Ullah et al. (2016) into a system of fractional order equations. These works were studied further by Lestari (2025) focusing on the numerical solutions to the model. When control to the smoking behavior is desired, optimal control model has been proposed by Ilmayasinta and Purnawan (2021).

The main challenge with modeling the problem into differential equations is how to solve the system accurately (Mungkasi, 2022b). Literature provides a number of methods to solve them, such as, the works of Ahmad and Charan (2016), Burden and Faires (2011), Butcher (2008), Hairer et al. (1987), as well as Hammachukiattiku et al. (2021). These methods produce approximate solutions numerically. Amongst those methods, the family of the Runge-Kutta methods is most popular (Hurit and Mungkasi, 2021).

Although a number of mathematical models have been proposed to describe the dynamics of smoking behavior, including the work of Ullah et al. (2016) and its extension by Günerhan et al. (2022), most of these studies mainly emphasize model formulation and qualitative analysis. The numerical simulations presented are generally used only to illustrate the behavior of the system, without further discussion on the accuracy of the numerical methods. In particular, aspects such as order of accuracy, convergence, and error analysis are rarely addressed, so the reliability of the numerical results is not fully examined.

This paper aims to verify the order of accuracy of the Runge-Kutta-Fehlberg numerical method when it is used to solve the mathematical model of the spread of smoking behavior. To do so, we recall the mathematical model and the computational procedure in the next section. Then, results and discussion are provided. Finally, we conclude the paper with several remarks.

METHOD

Mathematical Model

We recall the mathematical model of the spread of smoking behavior used by Ullah et al. (2016). The population consists of five subpopulations, namely, the subpopulations of potential smokers $P(t)$, occasional smokers $O(t)$, active smokers $S(t)$, temporary quitters $T(t)$, and permanent quitters $Q(t)$ at time t. The parameters involved in the model are the recruitment rate Λ, the effective contact rate β between potential smokers and active smokers, the rate α_1 at which occasional smokers become regular smokers, the contact rate α_2 between smokers and temporary quitters who revert back to smoking, and the natural death rate μ, the smoking-quitting rate γ, the fraction $(1-\delta)$ of smokers who temporary quit smoking (of the rate γ), and the remaining fraction δ of smoking who permanently quit smoking.

The dynamics of the system are described by the following system of five nonlinear ordinary differential equations (Ullah et al., 2016):

$$\frac{dP}{dt} = \Lambda - \beta PS - \mu P,$$

$$\frac{dO}{dt} = \beta PS - \alpha_1 O - \mu O,$$
$$\frac{dS}{dt} = \alpha_1 O + \alpha_2 ST - (\mu + \gamma)S,$$
$$\frac{dT}{dt} = \gamma(1-\delta)S - \alpha_2 ST - \mu T,$$
$$\frac{dQ}{dt} = \delta\gamma S - \mu Q.$$

The interactions between subpopulations are illustrated in the schematic diagram shown in Figure 1.

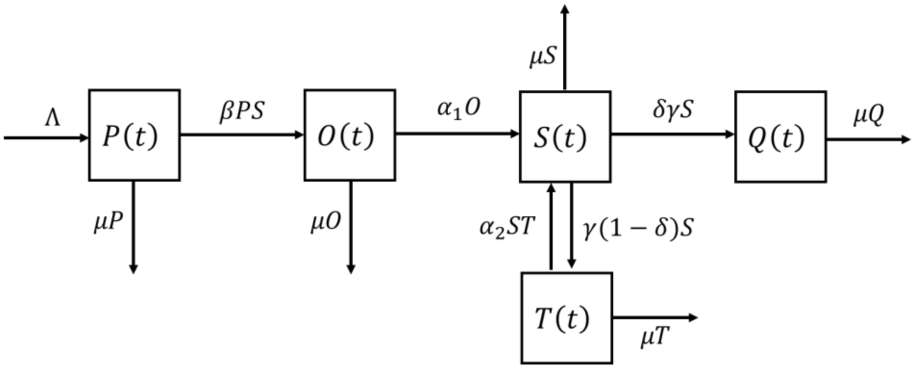


Figure 1. Schematic Diagram of the Model for the Spread of Smoking Behavior

We observe that the smoker-free equilibrium is $E_0 = (\Lambda/\mu, 0,0,0,0)$. This smoker-free equilibrium is locally asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$, where the reproduction number is

$$R_0 = \frac{\alpha_1 \beta}{(\mu + \gamma)(\alpha_1 + \mu)}.$$

We are interested in simulation towards this equilibrium, as if it could be possible, the population would be smoker-free in the long term of time.

Computational Procedure

The numerical method used to solve the mathematical model is the fifth order Runge-Kutta-Fehlberg (RKF5), as implemented by Lestari (2025). We refer readers to standard textbooks for numerical methods, such as, Burden and Faires (2011), Butcher (2008), as well as Mathews and Kurtis (2004). For the error analysis, we compare our results of RKF5 with the results of the ODE45 solver known in MATLAB, that is, the RK45 solver in Python. Our simulations are all conducted using the Python software. Figure 2 shows the flowchart for the numerical simulation of the spread of the smoking behavior using RKF5 and RK45 methods.

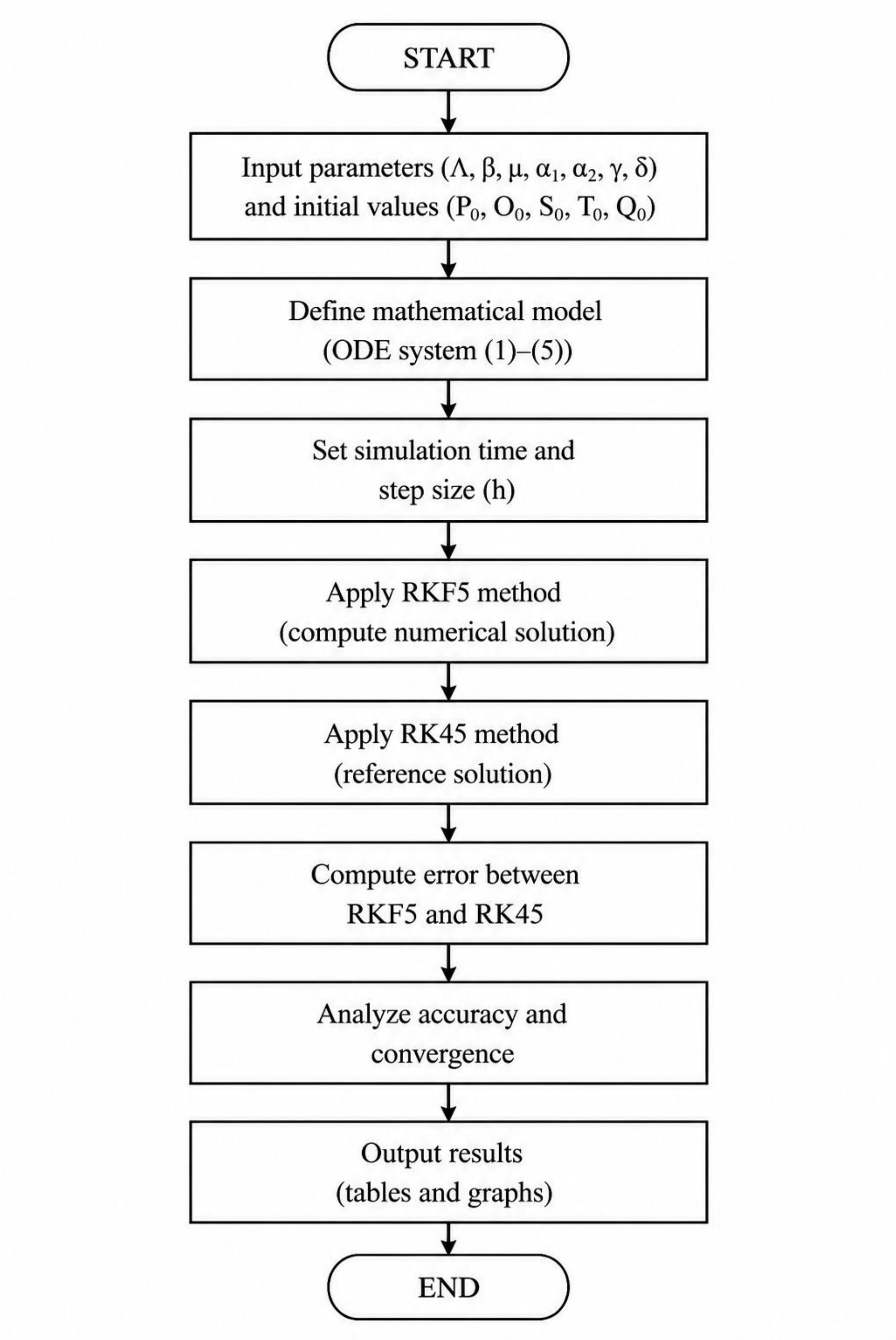


Figure 2. Flowchart for the Numerical Simulation of the Spread of the Smoking Behavior Using RKF5 and RK45 Methods

RESULTS AND DISCUSSION

Here we report our results in solving the mathematical model. The initial conditions are adapted from Ullah et al. (2016) and Günerhan et al. (2022). At $t = 0$, the initial population is given by $P_0 = 40$, $O_0 = 10$, $S_0 = 20$, $T_0 = 10$, and $Q_0 = 5$. The parameter values used in this model are as follows $\Lambda = 1$, $\beta = 0.14$, $\mu = 0.05$, $\alpha_1 = 0.002$, $\alpha_2 = 0.0025$, $\gamma = 0.8$, and $\delta = 0.1$. With these settings, we obtain the simulation results.

The dynamics of the subpopulations in terms of smoking behavior are shown in Figures 3 and 4. The dynamics mainly observable at around the beginning, that is, in the period $t \in [0,25]$ as in Figure 3. Even clearer, the dynamics can be seen for $t \in [0,2]$ as in Figure 4. As time evolves, subpopulations approach the equilibrium, which is realistic. This behavior is consistent with the condition determined by the value of the reproduction number R_0, where $R_0 < 1$ indicates that the smoking behavior will gradually decline and the system converges to a stable (smoker-free) equilibrium. See the work of Ullah et al. (2016) for the theory on the reproduction number R_0.

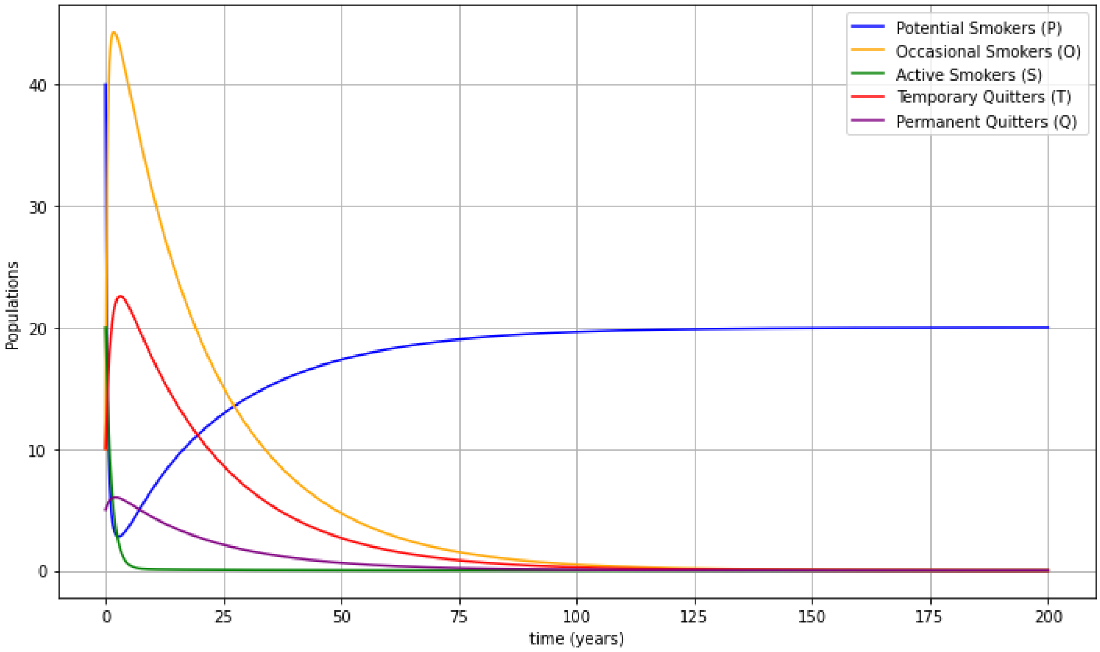
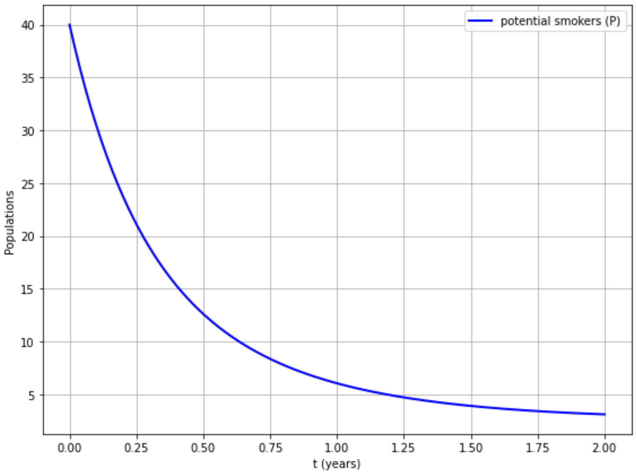
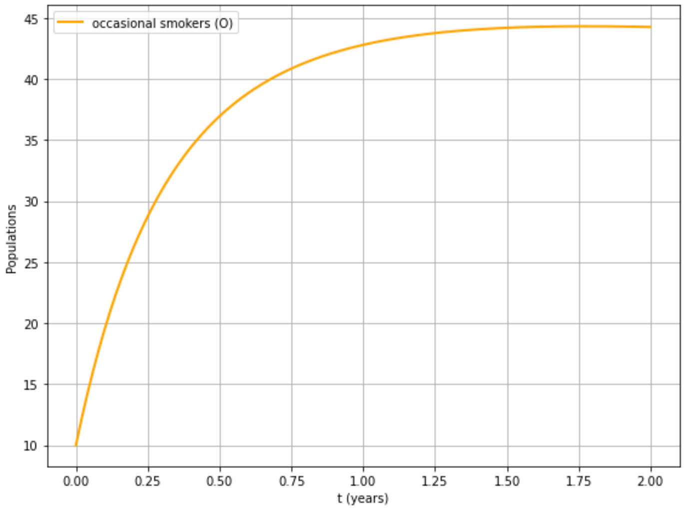


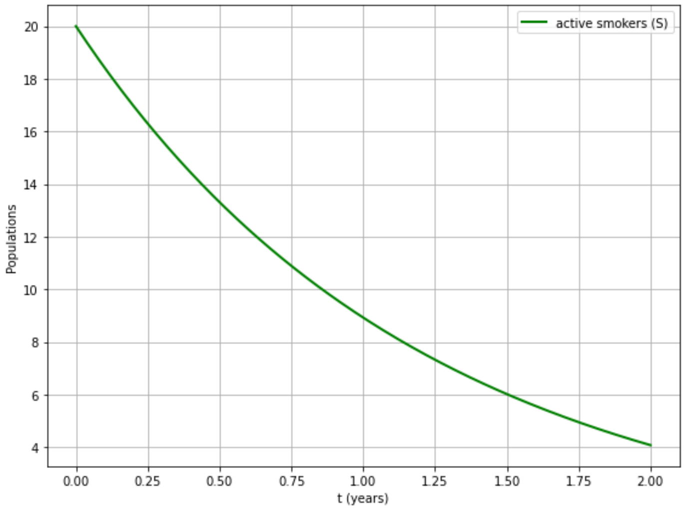
Figure 3. Simulation Results of the Mathematical Model of the Spread of Smoking Behavior Using the RKF5 Method



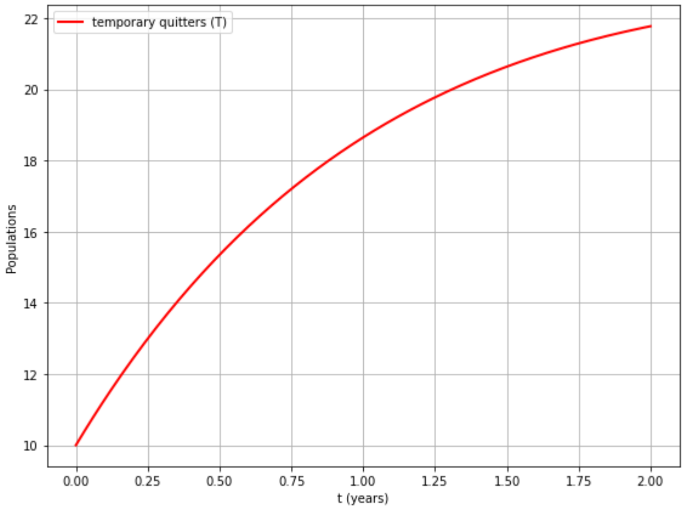
(a)


occasional smokers (O)
Populations
45
40
35
30
25
20
15
10
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t (years)

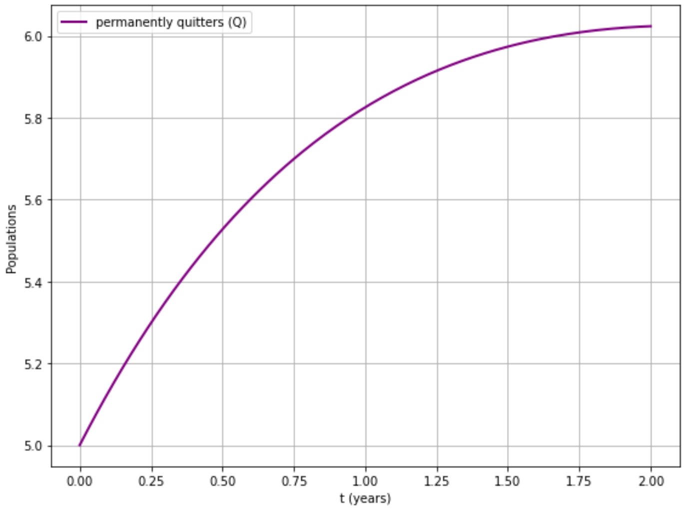
(b)


active smokers (S)
Populations
20
18
16
14
12
10
8
6
4
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t (years)

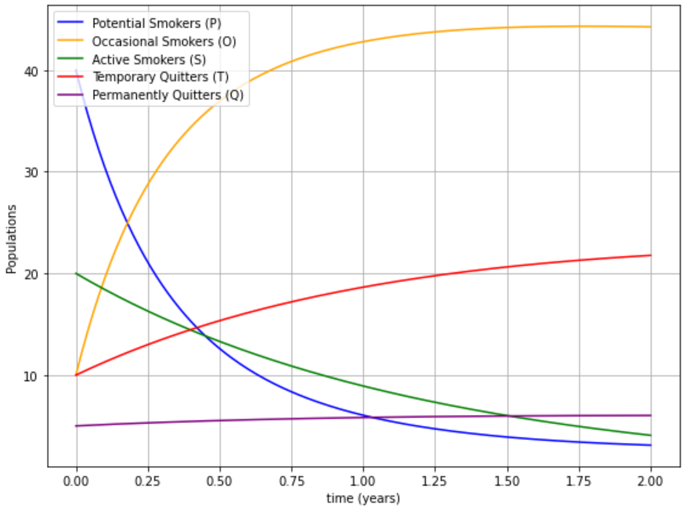
(c)


temporary quitters (T)
Populations
22
20
18
16
14
12
10
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t (years)

(d)



(e)



(f)

Figure 4. Solutions to Subpopulations for $t \in [0, 2]$:
(a) P, (b) O, (c) S, (d) T, (e) Q, (f) All

Subsequently, the accuracy of the smoking behavior spread model simulation using the RKF5 method is evaluated. The purpose of this simulation is to evaluate the accuracy and the order of the RKF5 method in solving the model of the spread of smoking behavior. The RKF5 solutions are compared to the RK45 solutions, where the RK45 method is provided in the SciPy library in the Python software. Simulations are conducted with several time step (h) variations, starting from $h = 0.2$, $h = 0.1$, and successively halving to smaller values. For each h, interpolation is performed at five time points ($t = 10, 20, 30, 40, 50$ years), and the results are compared to the reference RK45 solution set with an absolute error tolerance (a_{tol}) and relative error tolerance (r_{tol}) of 10^{-13}. The RK45 solution is chosen as the reference exact solution, since with such a small tolerance (such as 10^{-13}), the computed solution becomes very accurate and very close to the exact solution (Butcher, 2008; Hairer et al., 1987). Note that the default values in solve_ivp procedure in Python for rtol (relative tolerance) is 10^{-3} and atol (absolute tolerance) is 10^{-6}.

The absolute errors between the RKF5 and RK45 results are computed and averaged to obtain the mean error for each subpopulation. It can be observed that the error decreases significantly as the time step is reduced, indicating that the RKF5 method produces increasingly accurate solutions. This decreasing trend also indicates the convergence of the RKF5 method. Mungkasi (2022a) as well as Widi and Mungkasi (2025) stated that the order of accuracy can be evaluated using the following formula:

$$\text{Order of Accuracy} = \frac{\log\left(\frac{\text{error}_1}{\text{error}_2}\right)}{\log\left(\frac{h_1}{h_2}\right)} \tag{10}$$

Formula (10) is used to calculate the order of accuracy of the numerical solutions with respect to the varying time step. We record the average of absolute errors in Table 1, and the computational order of accuracy in Table 2. We obtain that the numerical order of accuracy approaches 5 as the time step tends to zero, which is consistent with the theoretical order of the RKF5 method. This confirms that the RKF5 method achieves its theoretical fifth-order accuracy.

Table 1. Average Error of Each Subpopulation

h	**Average Error**				
	P	O	S	T	Q
0.2	8.60×10^{-06}	7.95×10^{-06}	2.24×10^{-08}	5.60×10^{-07}	6.35×10^{-08}
0.1	9.92×10^{-08}	8.90×10^{-08}	2.75×10^{-10}	8.88×10^{-09}	1.02×10^{-09}
0.05	1.50×10^{-09}	1.28×10^{-09}	4.58×10^{-12}	1.90×10^{-10}	2.23×10^{-11}
0.025	2.78×10^{-11}	2.23×10^{-11}	1.21×10^{-13}	4.96×10^{-12}	6.02×10^{-13}

Table 2. Order of Accuracy of the Numerical Solutions for Each Subpopulations with Various Time-Step Values

h	**Solution Order of Accuracy**				
	P	O	S	T	Q
0.2 and 0.1	6.437406	6.480881	6.351554	5.978689	5.957877
0.1 and 0.05	6.050134	6.120125	5.907124	5.544514	5.517328
0.05 and 0.025	5.752988	5.843721	5.236673	5.262850	5.212113

We observe further that some computed orders of accuracy slightly exceed the theoretical order of 5, with values ranging from approximately 5.2 to 6.5. This phenomenon may occur due to numerical effects such as error cancellation or superconvergence, especially when the error values become very small. With these obtained results, RKF5 is promising to solve a wide range of systems of ordinary differential equations. These results confirm the reliability of the RKF5 method for solving nonlinear dynamical systems. This RKF5 should be able to solve other (various) mathematical models in the forms of systems of ordinary differential equations for infectious disease spreads, such as those researched by Mungkasi (2021), Tengger and Winarni (2022), Darajat (2023), and Winarni et al. (2024).

CONCLUSION

This study has solved the mathematical model of the spread of smoking behavior using the Runge-Kutta-Fehlberg method (RKF5). The numerical results show that the solutions converge toward the equilibrium, which is consistent with the theoretical stability of the system. In addition, the numerical results confirm that the RKF5 method achieves an order of accuracy close to five, in agreement with its theoretical order. This study is limited to numerical experiments and does not incorporate empirical data for validation. Future research direction could include empirical data so the validation of the mathematical model would be validated from reality.

ACKNOWLEDGMENT

This paper is developed from the masters thesis (Lestari, 2025) of the first author (Yuliastuti Dwi Lestari) under the supervision of the second author (Sudi Mungkasi) at Sanata Dharma University in 2025. The authors thank Sanata Dharma University for the support.

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