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HOMOMORPHISM ON ROUGH GROUPS

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ABSTRACT
Rough set is a mathematical concept to model uncertainty, vagueness, or incomplete information. This research discusses the concept and properties of homomorphisms on rough groups. Homomorphisms on rough groups are formed by mapping two upper approximations of subsets of classical groups, where those approximations are classical subgroups. The results show that a homomorphism on rough groups preserves the binary operation and a classical group homomorphisms is a special case of homomorphisms on rough groups. Furthermore, the properties of homomorphisms on rough groups related to preservation identity element and inverse element, as well as injective and surjective functions, are obtained. The concepts of kernel and image of homomorphisms on rough groups are defined by mapping elements within these approximations. In addition, a necessary and sufficient condition for a homomorphisms on rough groups to be isomorphisms is iff the kernel contains only the identity element of the domain and the image equals the entire codomain.

Keywords: Rough Sets, Rough Groups, Homomorphisms, Homomorphisms, Upper Approximations.

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PRELIMINARY

Rough set is a mathematical concept used to model uncertainty, vagueness, or incomplete information (El-Bably & El-Sayed, 2022), (Beydagi et al., 2026). Rough set theory was first developed by Polish scientist Zdzislaw Pawlak in 1982 (Ayuni et al., 2022), (Yanti & Fitriani, 2024). Unlike classical sets (crisp sets) whose membership is fixed, or fuzzy sets which use degrees of membership (values between 0 and 1), rough sets express uncertainty using boundary regions constructed from available information (Cahyo et al., 2025), (Prakash & Shukla, 2025), (Xu et al., 2025).

In classical group theory, a group is an algebraic structure consisting of a classical set equipped with a binary operation that satisfies four axioms. The concept of a group is an important foundation in modern mathematics, especially in abstract algebra. Discussions of classical groups can refer to (Judson, 2022), (Mayuri et al., 2024), (R. Kumar, 2022), (Westheimer, 2024) (S. A. Kumar, 2025). In classical group theory, a group homomorphism is a function that preserves binary operations on a group. Given two groups G and G', then the function $f: G \to G'$ is called a group homomorphism for every $x, y \in G$ satisfying $f(x * y) = f(x) *' f(y)$. In group homomorphism, the discussion of the kernel structure and image of a group homomorphism is an important study for the foundation of the Main Theorem of Group Homomorphisms. The discussion of group homomorphisms refers to (Rupnow & Sassman, 2022), (Caraquil & Baldado, 2023), (Cheng, 2023), (Chen & Lanier, 2024), (Zhang & Liu, 2026).

Group theory has been discussed on both fuzzy sets and rough sets (Choudhary & Biswas, 2020) (Agusfrianto et al., 2022). The formation of rough groups is done by combining two mathematical concepts, namely group theory and rough set theory (Altassan et al., 2020). The motivation for this formation is to make the uncertain data structure in rough sets into an algebraic system that has consistent operating rules, namely the group structure (Kieou et al., 2023) (Sangeetha & Sathish, 2023). In classical groups, the four requirements of the group axioms (closed, associative, identity, inverse) must be directly satisfied by each element in the set G, however, in rough groups the fulfillment of the four group axioms is satisfied in upper approximations of the set G (Alharbi et al., 2021), (Nugraha et al., 2022).

The difference in main characteristics between classical and rough groups lies in the stability of their boundaries (Ramanna et al., 2023). While classical groups tend to be rigid and deterministic, rough groups are more dynamic due to the inherent uncertainty of rough set theory (Demiralp, 2024). This difference in characteristics will undoubtedly affect the existence of identity elements, inverse elements, boundary regions, invariance properties with respect to relations, and the existence of subgroups in rough groups (Setyaningsih et al., 2021), (Hafifulloh et al., 2022), (Yasar, 2023) (Abd-Ellatif & Ramadan, 2025).

Nugraha et al. (2022) and Yasar (2023) have explained the formation of rough groups and several properties related to the structure of rough groups. The discussion of homomorphisms on rough groups has not been done. In this study, the author will develop the discussion of homomorphisms on rough groups based on the structure of the rough groups in and Nugraha et al. (2022) and Yasar (2023). The main motivation behind the formation of homomorphisms on rough groups is need to simplify, compare, and create data structures that contain uncertainty in rough groups without destroying algebraic relationships within them. Homomorphisms on rough groups are expected to be an algebraic solution to bridge these limitations.

METHODS

The research methodology used in this article employs a theoretical research approach with axiomatic logical reasoning to develop new concepts based on previous research related to rough groups. In the initial stage, a function is constructed that not only preserves algebraic binary operations but also compatible with equivalence relations (approximation spaces) on rough sets. Next, a mathematical analysis is performed to derive and prove the properties of the axiomatic deductive method. The steps are as follows.

1. Construct homomorphism on rough group with the following sub steps.
   a. Let U_1 and U_2 each be a classical group.
   b. Let G_1 be a subset of $U_1$and G_2 subset of U_2, then determine the upper approximation of $G_1$and G_2.
   c. Show that the upper approximations are classical subgroup
   d. Define a mapping between two upper approximations
   e. Show that the mapping preserves the binary operation
   f. Show that a classical group homomorphism is a special case of homomorphism on rough group
2. Construct the properties of homomorphism on rough group with the following sub steps.
   a. Show that homomorphism on rough group preserves the identity element
   b. Show that homomorphism on rough group preserves the inverse element
   c. Explain the definition of the kernel and image of homomorphism on rough group
   d. Explain the necessary and sufficient conditions for homomorphism on rough group using the concepts of kernel and image
   e. Show the injective and surjective properties of homomorphism on rough group.

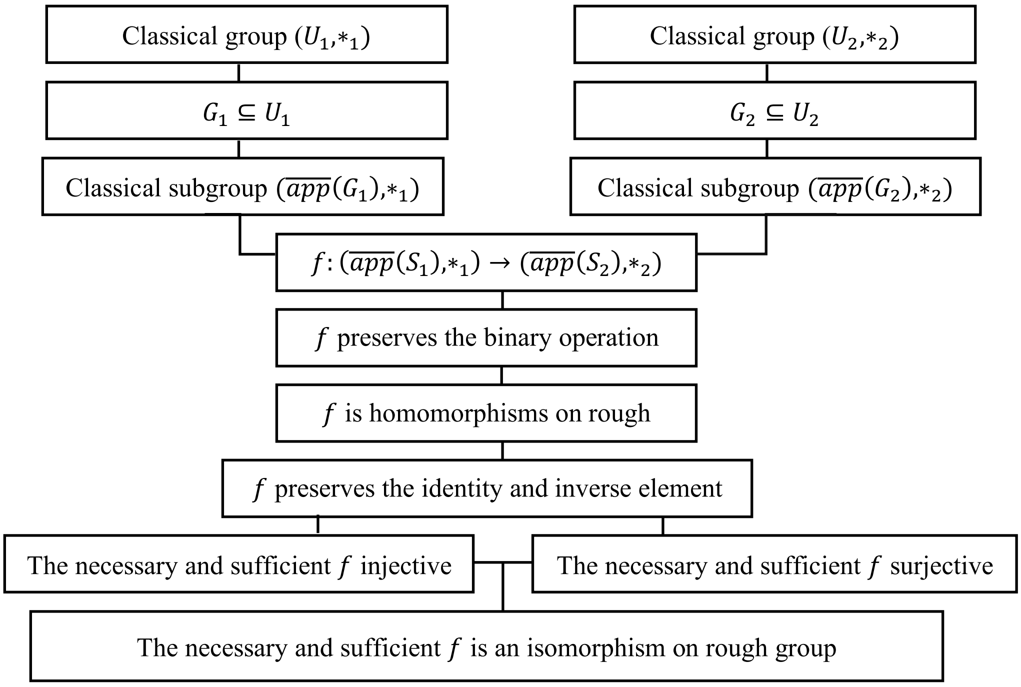

Figure 1. Procedure of Research

RESULT AND DISCUSSION

Definition of Rough Group

In classical algebraic structure, the concept of classical group are known as a mathematical system consisting of a pair of non-empty sets and a binary operation that satisfies the associative property, contains an identity element, and has an inverse of each element. By relating the concepts of rough set and classical group, the concept of rough group is obtained. The definition of a rough group is presented in Definition 1 below.

Definition 1. *Suppose the approximation space $(U, \mathcal{R})$ and the binary operation $*$ are defined on U, $(G,*)$ with $G \subseteq U$ is considered a rough group if the following axioms are satisfied:*

(i) *For all $x, y \in G$, $x * y \in \overline{app}(G)$*

(ii) *Association property holds in $\overline{app}(G)$*

(iii) *Exist $e \in \overline{app}(G)$ such that for all $x \in G$,*

$$x * e = x = e * x$$

e is referred to as rough group G's rough identity element.

(iv) *For all $x \in G$, exists $x^{-1} \in G$ such that,*

$$x * x^{-1} = e = x^{-1} * x$$

x^{-1} is referred to as rough group G's rough inverse element.

Notice that the classical group $(G,*)$ with $G = \overline{app}(G)$ satisfies all the axioms of a rough group. Therefore, a classical group can be regarded as a special case of a rough group. In other words, every classical group is a rough group. However, not every rough group is a classical group.

Definition 2. *A rough group* $(G,*)$ *is called a commutative rough group if* $\forall x, y \in G$, *we have* $x * y = y * x$.

Definition of Homomorphism on Rough Group

In classical groups theory, the concept of a classical group homomorphisms is known, which is a mapping between two classical groups while maintaining its binary operations. In the previous discussion, it has been shown that classical groups are a special case of rough groups. Therefore, the concept of a classical group homomorphism can be extended to include approximation structures, specifically upper approximations of a set as introduced by homomorphisms on rough groups. The definition of a homomorphism on a rough group is presented in Definition 3. below.

Definition 3. *Let* $(U_1, \mathcal{R}_1), (U_2, \mathcal{R}_2)$ *be two approximation spaces, and* $(U_1,*_1), (U_2,*_2)$ *be two classical groups. Let* $G_1 \subseteq U_1$ *and* $G_2 \subseteq U_2$, *such that* $\overline{app}(G_1)$ *and* $\overline{app}(G_2)$ *be two classical subgroups of* U_1 *and* U_2. *If* $f: (\overline{app}(G_1),*_1) \to (\overline{app}(G_2),*_2)$, *satisfies*

$$f(x *_1 y) = f(x) *_2 f(y)$$

$\forall x, y \in \overline{app}(G_1)$, *then* $f: (G_1,*_1) \to (G_2,*_2)$ *is a homomorphisms on rough groups.*

In other word, a homomorphisms on rough groups can be regarded as an extension of a classical group homomorphisms, i.e. a mapping that preserves the binary operation on the upper approximation of a subset of a classical group, where the upper approximation is a classical subgroup. The following is an example of homomorphisms on rough groups applied to the approximation space $\{S_3, \mathcal{R})$.

Example 4.

Let $\overline{app}(G_1) = \{(1), (123), (132)\}$ and $\overline{app}(G_2) = \{(1)\}$ be classical subgroups of $(S_3,\circ)$. Next, a mapping is defined

$$f: (\overline{app}(G_1),\circ) \to (\overline{app}(G_2),\circ) \text{ with } f(x) = (1), \quad \forall x \in \overline{app}(G_1)$$

Thus, we obtain

$$f(1) = f(123) = f(132) = (1).$$

Therefore, $\forall x, y \in \overline{app}(G_1)$ holds

$$f(x) \circ f(y) = (1) \circ (1) = (1) = f(x \circ y).$$

Therefore, $\forall x, y \in \overline{app}(G_1)$ holds $f(x \circ y) = f(x) \circ f(y)$. Furthermore, according to the Definition 3, we obtain that the mapping $f: (G_1, \circ) \to (G_2, \circ)$ is a homomorphisms on rough groups. ■

The Relationship between Classical Group Homomorphisms and Homomorphisms on Rough Groups

Based on Definition 3. and Example 4. we find that classical group homomorphisms is a special case of homomorphisms on rough groups when

$$\overline{app}(G_1) = G_1 \text{ and } \overline{app}(G_2) = G_2.$$

This shows that every classical group homomorphisms can be viewed as homomorphisms on rough groups in this special case. The statement does not apply vice versa, that is not every homomorphism on rough group is a classical group homomorphisms. Example 5. below shows a homomorphisms on rough groups that is not a classical group homomorphism.

Example 5.

Let $\overline{app}(X) = U = \mathbb{Z}$ with $X = \{-3, -2, -1, 0, 1, 2, 3\}$ and $(\mathbb{Z}, +)$ is a classical group. Next, a mapping $f: (\overline{app}(X), +) \to (\overline{app}(X), +)$ is defined by $f(x) = -x$. Note that $\forall x, y \in \overline{app}(X)$ holds

$$f(x + y) = -(x + y) = -x + (-y) = f(x) + f(y).$$

Based on Definition 8, since $\forall x, y \in \overline{app}(X)$ holds $f(x + y) = f(x) + f(y)$, then $f: (X, +) \to (X, +)$ is a homomorphism on rough group. However, $(X, +)$ is not a classical group because the $+$ operation is not closed on X i.e. there exists $3 \in X$ such that $3 + 3 = 6 \notin X$. Therefore, the mapping $f: (X, +) \to (X, +)$ is not a classical group homomorphism. Thus, not every homomorphism on rough group is a classical group homomorphism. ■

In addition, an example of a classical group homomorphisms that is not a homomorphisms on rough group is also given.

Example 6.

Let $U = \mathbb{Z}$ and $(\mathbb{Z}, +)$ is a classical group. Let $\overline{app}(G_1) = \mathbb{Z}$ with $G_1 = \{0\}$ and $(G_1, +)$ is a classical group. Since $G = \{0\}$, the only possible pair of elements is $x = y = 0$. Therefore, a mapping $f: (G_1, +) \to (G_1, +)$ defined by

$$f(n) = \begin{cases} 0, & n = 0 \\ 1, & n \neq 0 \end{cases}$$

is a classical group homomorphism because $f(0 + 0) = f(0) + f(0) = 0$ for all $x, y \in \overline{app}(G_1)$.

However, this f not a homomorphism on rough group because there exist $1, -1 \in \overline{app}(G_1)$ such that

$$f(1 + (-1)) = f(0) = 0 \text{ but } f(1) + f(-1) = 1 + 1 = 2.$$

Thus, not every classical group homomorphism is a homomorphism on rough group. ■

Properties of Homomorphism on Rough Group

Based on Definition 3., Lemma 7. below show that homomorphisms on rough groups preserves identity element and inverse element.

Lemma 7. *If a mapping $f: (G_1,*_1) \to (G_2,*_2)$ is a homomorphisms on rough groups, then*

(i) If e_1 is the rough identity element on $\overline{app}(G_1)$, then $f(e_1)$ is the rough identity element on $\overline{app}(G_2)$.

(ii) For all $x \in \overline{app}(G_1)$, $f(x^{-1}) = [f(x)]^{-1}$.

Based on Lemma 7. it is obtained that if a mapping f is homomorphisms on rough groups, then f preserves identity element, that is, the result of mapping the identity element on $\overline{app}(G_1)$ is the identity element on $\overline{app}(G_2)$. In addition, f also preserved the inverse element, that is $\forall x \in \overline{app}(G_1)$ holds that $f(x^{-1})$ is the inverse of $f(x)$.

Furthermore, the following definition explains the concepts of image and kernel in homomorphisms on rough groups.

Definition 8. *Suppose $f: (G_1,*_1) \to (G_2,*_2)$ is a homomorphisms on rough groups, then $\forall x \in \overline{app}(G_1)$ holds*

$$f(x *_1 y) = f(x) *_2 f(y).$$

Therefore, the kernel of f, denoted by $ker\ f$ is defined as

$$ker\ f = \{x \in \overline{app}(G_1) \mid f(x) = e_2\}$$

and image of f, denoted by $im\ f$ is defined as

$$im\ f = \{y \in \overline{app}(G_2) \mid y = f(x)\}.$$

Based on this definition, the concepts of kernel and image in homomorphisms on rough groups are extensions of the concepts of kernel and image in classical group homomorphisms, which apply to upper approximations of a set. Furthermore, a mapping is said to be surjective if every element in the codomain has a preimage and injective if every element only has one preimage. Note that the kernel of f contains all elements that map to the identity element in $\overline{app}(G_2)$ so f to be injective if the kernel only contains one element e_1. The sufficient and necessary conditions for f to be injective and surjective presented in Lemma 9. below.

Lemma 9. *If f is a homomorphisms on rough group, then* *(i)* f *to be injective if and only if* $\ker f = \{e_1\}$.

(ii) f *to be surjective if and only if* $\operatorname{im} f = \overline{app}(G_2)$,

Proof.

(i) (⇒) It is known that f is injective. Take any $x \in \ker f$, such that $f(x) = e_2$ holds. Since $f(x) = e_2$, we obtain $f(x) = f(e_1)$. Based on the injective function of f, $x = e_1$ is obtained. Thus, $\ker f = \{e_1\}$ is obtained.

(⇐) Assume that $\ker f = \{e_1\}$. Let $x, y \in \overline{app}(G_1)$. Suppose that $f(x) = f(y)$, then

$$f(x) *_2 f(y)^{-1} = e_2.$$

Since f is a homomophism,

$$f(x *_1 y^{-1}) = f(x) *_2 f(y^{-1}) = f(x) *_2 f(y)^{-1} = e_2.$$

Hence, $x *_1 y^{-1} \in \ker f$. Since $\ker f = \{e_1\}$ then it must be $x = y$. Therefore, f injective. ■

(ii) (⇒) It is known that f is surjective. By the definition of surjectivity, for every $y \in \overline{app}(G_2)$, there exist an element $x \in \overline{app}(G_1)$ such that $f(x) = y$. Since

$$\operatorname{im} f = \{y \in \overline{app}(G_2) \mid y = f(x)\},$$

every element of $\overline{app}(G_2)$ belongs to $\operatorname{im} f$. Therefore $\operatorname{im} f = \overline{app}(G_2)$.

(⇐) It is known that $\operatorname{im} f = \overline{app}(G_2)$. Let $y \in \overline{app}(G_2)$, by the definition of the image, there exist $x \in \overline{app}(G_1)$ such that $f(x) = y$. Therefore, f surjective. ■

The following definition of an isomorphisms on rough groups is based on Lemma 9.

Definition 10. *A homomorphism on rough groups is called an isomorphism if it is both injective and surjective.*

CONCLUSION

Homomorphism on rough group is a mapping that preserves the binary operation on the upper approximations of subsets of classical group, where those approximations are classical subgroup. A classical group homomorphism can be said to be a special case of homomorphism on rough group. Future research may extend the homomorphism concept to other rough algebraic structures.

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