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SOME COINCIDENCE POINT THEOREMS IN MODULAR SPACES

Afifah Hayati
Universitas Nahdlatul Ulama Purwokerto, Jln. Sultan Ageng No.42, Karangklesem, Purwokerto Selatan, Purwokerto, Jawa Tengah, afifahhayati.mail@gmail.com

ABSTRACT

In metric spaces, defined a generalization of contraction mappings, called quasi-contraction mappings, that satisfy a condition that states there exists a nonnegative real number which is less than one such that for any two points, the metric of the image of that mappings at the two points is less than or equal to the real number multiplied by the maximum of the metric of the two points, the metric of each point with the image of the point itself, and the metric of each one point with the image of the other point. Then, defined Suzuki-contraction mappings that satisfy if two points satisfy half of the metric of one point with the image of the mappings at the point itself, then the metric of the image of the two points is less than the metric of the two points. A modular space is a vector space equipped with a modular that is the generalization of norm. Therefore, studying the definition and the properties of modular spaces as well as the definition of quasi-contraction mappings and Suzuki-contraction mappings in modular spaces, in this article, we shall show that coincidence point theorem for quasi-contraction mapping in modular spaces is a generalization of the fixed point. Furthermore, we shall show that fixed point theorem and coincidence point theorem for Suzuki-contraction mappings in modular spaces is generalization of the fixed point theorem and the coincidence point theorem in metric spaces with some extra assumptions.

Keywords : *Coincidence Point, Quasi-Contraction Mappings, Suzuki-Contraction Mappings, Modular Space*

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PRELIMINARY

Today, the generalization of fixed point theorem is growing rapidly because this theorem has quite a number of applications. Most of these generalizations are the generalizations of the Banach Contraction Theorem, which states that if (X, d) is a complete metric space and $T: X \to X$ is a contraction mapping, then T has a unique fixed point. A point in the nonempty set X is said to be a coincidence point of two mappings defined on X if the image of the element in one of the mappings is equal to the image of the element in the other mapping. In 1998, Jungck & Rhoades introduced the definition of two mappings that are weakly-compatible in metric spaces, i.e. two mappings whose the composition of the mappings is commutative at their coincidence point. One of the generalizations of fixed point theorem is coincidence point theorem which involves two mappings. In general, the coincidence point theorem is obtained by providing one of the mappings satisfies the assumption of the fixed theorem on the range of the other mapping. Furthermore, the weakly-compatible condition satisfied by the two mappings guarantee the uniqueness of their coincidence point.

Both in studying the theory of fixed points and coincidence points, several concepts in metric spaces and normed spaces are needed as described by Berberian (1961), Kreyszig (1978), Rudin (1987) and Royden & Fitzpatrick (2010). Musielak (1983) introduced a functional that is more general than norm called modular. In addition, H. Rubin and J.E. Rubin (1985) gave several equivalences of the Axiom of Choice which are used in the discussion of coincidence point theorem. In addition, Haghi, Rezapour, dan Shahzad (2010) give a lemma that is the corollary of one of the equivalences of the Axiom of Choice, which is also used in the discussion of coincidence point theorem. In 1974, Ćirić introduced a mapping that is more general than contraction mappings in metric spaces known as quasi-contraction mappings, i.e. an itself mapping $T: X \to X$ such that (X, d) be a metric space and T satisfies a condition that states there exists $k \in [0,1)$ such that for all x and y in the metric space, one has $d\big(T(x), T(y)\big)$ is less than or equal to $k \max\{d(x,y), d\big(x, T(x)\big), d\big(y, T(y)\big), d\big(x, T(y)\big), d\big(y, T(x)\big)\}$. Moreover, Ćirić (1974) also introduced a fixed point theorem that is a generalization of Banach Contraction Theorem. Then, Khamsi (2008) introduced quasi-contraction mappings in modular spaces and the fixed point theorem for the mappings. Furthermore, Tomonari Suzuki (2009) introduced a mapping that satisfies a contraction condition on a metric space, i.e. an itself mapping T on the metric space (X, d) that satisfies for x and y in those metric spaces such that $\frac{1}{2}\, d\big(x, T(x)\big)$ is less than $d(x, y)$ implies $d\big(T(x), T(y)\big)$ is less than $d(x, y)$. The mapping that satisfies the conditions is known as Suzuki-contraction mappings. In addition, Tomonari Suzuki (2009) also gave a fixed point theorem for the mappings, that stated if (X, d) be a compact metric space and $T: X \to X$ be a Suzuki-contraction mapping, then T has a unique fixed point. Furthermore, Singh and Mishra (2010) extended the fixed point theorem for Suzuki-contraction mapping in metric spaces, that stated if (X, d) be a metric space, Y be a nonempty set, $T, S: Y \to X$ such that $T(Y) \subseteq S(Y)$ and $S(Y) \subseteq X$ be a compact set, and T be a Suzuki-contraction mapping on $S(Y)$ that satisfies for all $x, y \in X$ such that $T(x) = T(y)$ implies $S(x) = S(y)$, then T and S have a coincidence point. Moreover, if $Y = X$, T and S are weakly-compatible, then T and S have a unique common fixed point.

The development of the fixed point theorem for Suzuki-contraction mapping on metric spaces is described by Kumam, Gopal, & Budhia (2017), Padcharoen, Kumam, Saipara, & Chaipunya (2018), Alqahtani, Bindu & Karapinar (2019), Girgin & Öztürk (2019), Uddin, Perveen, Işik, & Bhardwaj (2021), and Beg, Mani, & Gnanaprakasam (2021). Then, some developments of the coincidence point theorem are given by Sumalai, Kumam, Cho, & Padchareon (2017) and Padcharoen, Kumam, & Gopal (2017). Moreover, the development of fixed point theorem for quasi-contraction mapping in metric spaces is given by Bisht, R. K. (2019) and Popescu & Stan (2019).

Khamsi (2008) only gave the fixed point theorem for quasi-contraction mapping in modular spaces. Therefore, in this article, we introduce the coincidende point theorem for quasi-contraction mapping in a modular space. In addition, since a modular is a functional that is more general than norm and the norm itself can be viewed as a metric by defining the metric of two points as the norm of the difference of the two points, then using the fixed point theorem for Suzuki-contraction mapping in metric spaces given by Tomonari Suzuki (2009) and the extended of the fixed point theorem for Suzuki-contraction mapping in metric spaces given by Singh and Mishra (2010), we also introduce the fixed point theorem and the coincidence point theorem for mapping Suzuki-contraction in modular spaces.

METHOD

The research method in this article is by study some literatures that begins with studying the literatures on research related to modular spaces, quasi-contraction mappings and Suzuki-contraction mappings in metric spaces which are the basis for studying quasi-contraction mappings and Suzuki-contraction mappings in modular spaces. Then, we study the fixed point theorem for quasi-contraction mapping and Suzuki-contraction mapping in metric spaces that is the basis for studying the fixed point theorem and the coincidence point theorem for the mapping in modular spaces.

To prove the theorems in this article, we use the following methods.

1. Direct Proving Method

   This method is usually applied to prove the theorem in the implication form $p \Rightarrow q$. The premise p as a hypothesis is used as an assumption. Next, using the premise p, it is shown that q holds.

2. Proving Method by Contradiction

   In this method, to prove the implication $p \Rightarrow q$, the first step is to assume $\sim q$, which is the negation of q, which implies a contradiction, that is, there are one or more contradictory statements with p.

3. Proving Method of Uniqueness

   To prove the uniqueness of an object, first, we show the existence of an object, say the object is x. There are two approaches that can be taken to prove that x is a unique object.

   i. Let y be any object, we will show that $y = x$.

   ii. Suppose y be any object such that $x \neq y$, we will show that there is one or more contradiction. This method uses the proving method by contradiction.

4. Proving Method by Induction

   The principle of induction is the inference of a statement related to n such that the n running on set of integers, which can be the set of natural numbers $\mathbb{N}$ or the subset of $\mathbb{N}$. In general, the statement related to natural number n can be written as $P(n)$. The following steps is to prove $P(n)$ holds, for all $n \in \mathbb{N}$.

   Step (i) : Prove that $P(n)$ holds, for $n = 1$.

   Step (ii) : Assume that $P(n)$ holds, for $n = k$. Then, we prove $P(n)$ holds, for $n = k + 1$.

   If $P(n)$ can be proven, for $n = 1$ in Step (i), then according to Step (ii), we can imply that $P(n)$ holds, for $n = 2$ that also implies $P(n)$ holds, for $n = 3$ and so on. Therefore, if Step (i) and (ii) can be proven, then we can conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

The research steps in this article are as follows:

1. Give the definition of the modular space and its properties,
2. Provide the definition of quasi-contraction mapping and Suzuki-contraction mapping in modular spaces,
3. Provide the fixed point theorem for quasi-contraction mapping which is proven by direct proving method which in the proving process, we also use the proving method by induction and uniqueness,
4. Give the coincidence point theorem for quasi-contraction mapping on modular spaces which is proved by the direct proving method,

5. Introduce the fixed point theorem for Suzuki-contraction mapping on modular spaces which is proven by the direct proving method, which in the process we use the proving method by contradiction and uniqueness, and
6. Introduce the coincidence point theorem for Suzuki-contraction mapping on modular spaces which is proved using the direct proving method.

RESULT AND DISCUSSION

In this section, we state and prove our main result. First, we give the definition of modular spaces.

Definition 1. Let X be a vector space over $\mathbb{R}$. A functional $\rho: X \to [0, \infty]$ is said to be a modular if for all $x, y \in X$, there hold the following:

(D1) $\rho(x) = 0$ if only if $x = \theta$ where θ is a null vector in X,

(D2) $\rho(-x) = \rho(x)$, and

(D3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

If we replace (D3) by

(D3') $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$,

then the modular ρ is said to be convex.

A vector space X equipped with modular ρ, denoted by (X, ρ), is said to be a modular space. One of the important subspaces in modular spaces (X, ρ) is

$$X_\rho = \left\{x \in X \mid \lim_{\lambda \to 0} \rho(\lambda x) = 0\right\}.$$

Furthermore, X_ρ is also modular spaces with the modular ρ. In the following discussion, what we called by modular spaces is X_ρ.

In the metric spaces, there are some topological concepts, such as compact set and the convergence of a sequence. Therefore, we will give some topological concepts in modular spaces, such as a convergent sequence and a Cauchy sequence in modular spaces, a complete modular space, and a compact set in modular spaces.

Definition 2. Let X_ρ be a modular space.

(i) The sequence $\{x_n\} \subseteq X_\rho$ is said to ρ-convergent to $x \in X_\rho$ if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ that satisfies $n \geq n_0$, one has $\rho(x_n - x) < \varepsilon$. Then, x is said to be a ρ-limit of $\{x_n\}$ and the sequence $\{x_n\}$ that ρ-convergent to x can be written by $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii) The sequence $\{x_n\} \subseteq X_\rho$ is said to be a ρ-Cauchy sequence if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ that satisfies $n, m \geq n_0$, one has $\rho(x_n - x_m) < \varepsilon$. Then, the sequence $\{x_n\}$ that is a ρ-Cauchy sequence can be written by $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(iii) A subset $C \subseteq X_\rho$ is said to be ρ-complete if each ρ-Cauchy sequence in C is ρ-convergent and its ρ-limit is in C.

(iv) A subset $C \subseteq X_\rho$ is said to be ρ-compact if each sequence in C has a subsequence that ρ-convergent.

Beside the sequence concept in modular spaces, we give the definition of bounded set on modular spaces. Then, we also give the definition of orbit and diameter of an orbit in modular spaces.

Definition 3. Let X_ρ be a modular space, $C \subseteq X_\rho$ be a nonempty set, and $T: C \to C$ be a self-mapping.

(i) A subset $C \subseteq X_\rho$ is said to be ρ-bounded if

$\delta_\rho(C) = \sup\{\rho(x - y) \mid x, y \in C\} < \infty$.

(ii) For any $n \in \mathbb{N}$, T^n with the formula

$$T^n(x) = (T \circ T \circ \ldots \circ T)(x), \text{ for all } x \in X_\rho$$

is a n times compositions of T and $T^0(x) = x$. Furthermore, T^n is a mapping.

(iii) For any $x \in C$, define the orbit

$O(x) = \{x, T(x), T^2(x), \ldots\}$

and its ρ-diameter by

$\delta_\rho(x) = \mathrm{diam}\big(O(x)\big) = \sup\big\{\rho\big(T^n(x) - T^m(x)\big) \mid n, m \in \mathbb{N}\big\}$.

In 2008, Khamsi introduced a quasi-contraction mapping in modular spaces.

Definition 4. Let X_ρ be a modular space and $C \subseteq X_\rho$ be a nonempty set. The self-mapping $T: C \to C$ is said to be a ρ-quasi-contraction mapping if there exists $\alpha \in [0,1)$ such that for all $x, y \in C$, one has

$$\rho\big(T(x) - T(y)\big) \leq k \max \left\{ \begin{matrix} \rho(x - y), \rho\big(x - T(x)\big), \rho\big(y - T(y)\big), \\ \rho\big(x - T(y)\big), \rho\big(y - T(x)\big) \end{matrix} \right\}.$$

In the discussion about coincidence point theorem for quasi-contraction mappings in modular spaces, we need the fixed point theorem for quasi-contraction mappings in modular spaces. In addition, we also need a property that satisfies a modular, which is known as the Fatou property.

Definition 5. The modular ρ on X_ρ is said to has the Fatou property if for each $\{x_n\} \subseteq X_\rho$ that ρ-convergent to $x \in X_\rho$ implies

$\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$.

In 2009, Tomonari Suzuki introduced a mapping that satisfy a contraction condition in metric spaces that we called a Suzuki-contraction mapping. Therefore, we introduce the concept of Suzuki-contraction mappings in modular spaces.

Definition 6. Let X_ρ be a modular space dan $C \subseteq X_\rho$ be a nonempty set. The self-mapping $T: C \to C$ is said to be a ρ-Suzuki-contraction mapping if for all $x, y \in C$ such that

$$\frac{1}{2}\rho\big(x - T(x)\big) < \rho(x - y) \text{ implies } \rho\big(T(x) - T(y)\big) < \rho(x - y).$$

In the discussion about fixed point theorem and coincidence point theorem for ρ-Suzuki-contraction in modular spaces, we need a property of the modular ρ that is called Δ_2-type condition.

Definition 7. Let X_ρ be a modular space equipped with a modular ρ. The modular ρ is said to satisfies the Δ_2-type condition if there exists $K > 0$ such that for all $x \in X$ one has $\rho(2x) \leq K\rho(x)$.

Besides introduced quasi-contraction mappings in modular spaces, Khamsi (2008) give the fixed point theorem for quasi-contraction mappings in modular spaces without Δ_2-type condition.

Theorem 8. Let X_ρ be a modular space such that the modular ρ satisfies Fatou property, $C \subset X_\rho$ be a ρ-complete nonempty set, and $T: C \to C$ be a ρ-quasi-contraction mapping such that $\rho\big(x - T(x)\big) < \infty$ and $\rho\big(x - T(y)\big) < \infty$ for all $x, y \in C$. If $x \in C$ such that $\delta_\rho(x) < \infty$, then the ρ-limit w of $\{T^n(x)\}$ is a fixed point of T, that is $T(w) = w$. Moreover, if w^* is any fixed point of T in C such that $\rho(w - w^*) < \infty$, then one has $w = w^*$.

Proof. Since $T: C \to C$ ρ-quasi-contraction mappings, then there exists $k \in [0,1)$ such that for all $x, y \in C$, we have

$$\rho\big(T^n(x) - T^m(y)\big) \leq k \max\left\{\begin{array}{c}\rho\big(T^{n-1}(x) - T^{m-1}(y)\big), \rho\big(T^{n-1}(x) - T^n(x)\big), \\ \rho\big(T^m(y) - T^{m-1}(y)\big), \rho\big(T^{n-1}(x) - T^m(y)\big), \\ \rho\big(T^n(x) - T^{m-1}(y)\big)\end{array}\right\}.$$

This obviously implies the following result

$\delta_\rho(T^n(x)) \le k\, \delta_\rho(T^{n-1}(x))$.

Moreover, for all $n \ge 1$ and $m \in \mathbb{N}$, we have

$\rho(T^n(x) - T^{n+m}(x)) \le \delta_\rho(T^n(x)) \le k^n \delta_\rho(x)$.

Furthermore, since $\delta_\rho(x) < \infty$ and $k \in [0,1)$, then for all $n \ge 1$ and $m \in \mathbb{N}$ as $n, m \to \infty$, we have

$\rho(T^n(x) - T^{n+m}(x)) \to 0$.

Therefore, $\{T^n(x)\}$ is a ρ-Cauchy sequence. Using the assumption that C is ρ-complete, there exists $w \in C$ such that $\{T^n(x)\}$ ρ-convergent to w. Since for all $n \ge 1$ and $m \in \mathbb{N}$, we have

$\rho(T^n(x) - T^{n+m}(x)) \le k^n \delta_\rho(x)$

and ρ satisfies Fatou property, then we obtain

$\rho(T^n(x) - w) \le k^n \delta_\rho(x)$

as $m \to \infty$. Moreover, since T is ρ-quasi-contraction mappings, we get

$$\rho(T(x) - T(w)) \le k \max\{\rho(x - w), \rho(x - T(x)), \rho(T(w) - w), \rho(T(x) - w), \rho(x - T(w))\}$$

so that

$\rho(T(x) - T(w)) \le k \max\{\delta_\rho(x), \rho(w - T(w)), \rho(x - T(w))\}$.

By induction, we have

$\rho(T^n(x) - T(w)) \le \max\{k^n \delta_\rho(x), k\rho(w - T(w)), k^n \rho(x - T(w))\}$,

for all $n \ge 1$. Therefore, we have

$\limsup_{n\to\infty} \rho(T^n(x) - T(w)) \le k\, \rho(w - T(w))$.

Using the Fatou property satisfied by ρ, we get

$\rho(w - T(w)) \le \liminf_{n\to\infty} \rho(T^n(x) - T(w)) \le k\, \rho(w - T(w))$.

Since $k < 1$, then $\rho(w - T(w)) = 0$ or $T(w) = w$. So, it is true that T has a fixed point, that is w. Furthermore, let w^* be any fixed point of T such that $\rho(w - w^*) < \infty$, then $T(w^*) = w^*$ so that

$\rho(w - w^*) = \rho(T(w) - T(w^*)) \le k\rho(w - w^*)$.

Since $k < 1$, we get $\rho(w - w^*) = 0$ or $w = w^*$. ■

In the discussion about coincidence point theorem in modular spaces, we need a concept in set theory, that is Axiom of Choice.

Axiom of Choice. For each mapping F, there exists a mapping f such that $F(x) \neq \emptyset$ implies $f(x) \in F(x)$, for all x in the domain of F.

Based on the Axiom of Choice, for each nonempty set X and Y as well as a mapping $F: Y \to 2^X$, there exists a mapping $f: Y \to X$ such that for all $y \in Y$ one has $f(y) \in F(y)$.

One of the important properties of the mapping in Axiom of Choice is stated in the following theorem.

Theorem 9. Let X be a nonempty set. If $f: X \to X$ is a mapping, then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f: E \to X$ is an injective mapping.

Proof. Consider the set $f(X) = \{y = f(x) | x \in X\}$, we give a mapping $F: f(X) \to 2^X$ defined by

$$F(y) = \{x \in X \mid f(x) = y\}, \text{ for all } y \in f(X).$$

Using the Axiom of Choice, there exists $g: f(X) \to X$ be a mapping such that

$$g(y) \in F(y), \text{ for all } y \in f(X).$$

Since $g(y) \in F(y)$, there exits $x \in X$ such that $x = g(y)$ and $f(x) = y$. Therefore, we have

$$f\big(g(y)\big) = f(x) = y, \text{ for all } y \in f(X).$$

Now, put $E = \{g(y) \mid y \in f(X)\}$. First, we will proof that $E \subseteq X$. Let $g(y)$ be arbitrary element in E such that $y \in f(X)$, then $g(y) \in F(y)$ such that there exists $x \in X$ such that $g(y) = x$. Since $x \in X$ and $g(y) = x$, we get $g(y) \in X$. Therefore, for each $g(y) \in E$, we have $g(y) \in X$. So, it is true that $E \subseteq X$. Next, we will proof that $f: E \to X$ is an injective mapping. Let $g(y_1), g(y_2) \in E$ be given such that $y_1, y_2 \in f(X)$ and $f\big(g(y_1)\big) = f\big(g(y_2)\big)$. Since $y_1, y_2 \in f(X)$, we obtain $g(y_1), g(y_2) \in F(y)$ so that there exist $x_1, x_2 \in X$ such that

$$x_1 = g(y_1), x_2 = g(y_2) \text{ and } f(x_1) = y_1, f(x_2) = y_2.$$

Therefore, we get

$$f\big(g(y_1)\big) = f\big(g(y_2)\big) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow y_1 = y_2.$$

Thus, we have $g(y_1) = g(y_2)$. So, it is true that f is injective on E. Last, we have to proof $f(E) = f(X)$. Since $E \subseteq X$, then it is clear that $f(E) \subseteq f(X)$. Let $y \in f(X)$ be given, then we have $g(y) \in E$. Since $y \in f(X)$, we get $g(y) \in F(y)$ so that there exists $x \in X$ such that $y = f(x)$ and $x = g(y)$. Therefore, we obtain

$$y = f(x) = f\big(g(y)\big) \in f(E).$$

In other words, for any $y \in f(X)$ implies $y \in f(E)$. So, it is true that $f(X) \subseteq f(E)$. Therefore, we have $f(E) = f(X)$. ■

The following theorem state the coincidence point theorem for ρ-quasi-contraction mapping in modular spaces which is a generalization of the fixed point theorem.

Theorem 10. Let X_ρ be a modular space such that its modular ρ satisfies Fatou property, $C \subseteq X_\rho$ be a ρ-complete nonempty set, $T, S: C \to C$ be such that $\rho\big(S(x) - T(x)\big) < \infty$ and $\rho\big(S(x) - T(y)\big) < \infty$, for all $x, y \in C$. Assume that there exists $k \in [0,1)$ such that for all $x, y \in C$ one has

$$\rho\big(T(x) - T(y)\big) \leq k \max \left\{ \begin{matrix} \rho\big(S(x) - S(y)\big); \rho\big(S(x) - T(x)\big); \\ \rho\big(S(y) - T(y)\big); \rho\big(S(x) - T(y)\big); \\ \rho\big(S(y) - T(x)\big) \end{matrix} \right\}.$$

If $x \in C$ such that $\delta_\rho\big(S(x)\big) < \infty$, then there exists $\bar{x} \in C$ such that $T(\bar{x}) = S(\bar{x})$ or $\bar{x}$ is a coincidence point of T and S. Moreover, if $\bar{x}^*$ be any coincidence point of T and S as well as T and S are weakly-compatible such that $\rho\big(T(\bar{x}) - T(\bar{x}^*)\big) < \infty$, then T and S have a unique common fixed point.

Proof. According to Theorem 9, there exists $E \subseteq C$ such that $S(E) = S(C)$ and $S: E \to C$ is injective. Then, we define $U: S(E) \to S(E)$ such that

$$U\big(S(x)\big) = T(x), \text{ for all } S(x) \in S(E).$$

We will proof that U is a mapping. Let $S(x), S(y) \in S(E)$ be given such that $S(x) = S(y)$, we will show that $U\big(S(x)\big) = U\big(S(y)\big)$. Since $S(x), S(y) \in S(E)$, then we get $x, y \in E$. Furthermore, since S is injective and $S(x) = S(y)$, then we have $x = y$. Note that $E \subseteq C$ and T is a mapping on C. Hence, we obtain for any $x, y \in E$ such that $x = y$ implies

$$T(x) = T(y) \text{ or } U\big(S(x)\big) = U\big(S(y)\big).$$

Therefore, we get for all $S(x), S(y) \in S(E)$ such that $S(x) = S(y)$ implies $U\big(S(x)\big) = U\big(S(y)\big)$.

So, it is true that U is a mapping on $S(E)$. Since for all $S(x), S(y) \in S(E)$, we have

$$\rho\Big(U\big(S(x)\big) - U\big(S(y)\big)\Big) = \rho\big(T(x) - T(y)\big)$$

$$\leq k.\max \left\{ \begin{matrix} \rho\big(S(x) - S(y)\big); \rho\big(S(x) - T(x)\big); \\ \rho\big(S(y) - T(y)\big); \rho\big(S(x) - T(y)\big); \\ \rho\big(S(y) - T(x)\big) \end{matrix} \right\}.$$

Thus, U is a ρ-quasi-contraction mapping on $S(E)$. Moreover, since $\rho\big(S(x) - T(x)\big) < \infty$ and $\rho\big(S(x) - T(y)\big) < \infty$, for all $x, y \in C$, then we obtain

$$\rho\left(S(x)-U\big(S(x)\big)\right)<\infty \text{ and } \rho\left(S(x)-U\big(S(y)\big)\right)<\infty, \text{ for all } S(x),S(y)\in S(E).$$

Furthermore, since $S(x)\in S(C)$ such that $\delta_\rho\big(S(x)\big)<\infty$ and $S(E)=S(C)$, we have

$$S(x)\in S(E).$$

Therefore, since X_ρ is a modular space such that its modular ρ satisfies Fatou property, $S(E)=S(C)\subseteq C$ is ρ-complete such that $\rho\left(S(x)-U\big(S(x)\big)\right)<\infty$ and $\rho\left(S(x)-U\big(S(y)\big)\right)<\infty$, for all $S(x),S(y)\in S(E)$ and U is a ρ-quasi-contraction mapping on $S(E)$ as well as $S(x)\in S(E)$ such that $\delta_\rho\big(S(x)\big)<\infty$, then according to Theorem 8, there exists $S(\bar{x})\in S(E)$ such that $U\big(S(\bar{x})\big)=S(\bar{x})$ or $T(\bar{x})=S(\bar{x})$. So, $\bar{x}$ is a coincidence point of T and S. If $\bar{x}^*$ be any coincidence point of T and S as well as T and S are weakly-compatible such that $\rho\big(T(\bar{x})-T(\bar{x}^*)\big)<\infty$, we will proof T and S have a unique common fixed point. Since $\bar{x}$ and $\bar{x}^*$ are two coincidence points of T and S, then

$$T(\bar{x})=S(\bar{x}) \text{ dan } T(\bar{x}^*)=S(\bar{x}^*).$$

Note that $\rho\big(T(\bar{x})-T(\bar{x}^*)\big)<\infty$, we get

$$\rho\left(U\big(S(\bar{x})\big)-U\big(S(\bar{x}^*)\big)\right)<\infty.$$

Since $U\big(S(\bar{x})\big)=T(\bar{x})$, $U\big(S(\bar{x}^*)\big)=T(\bar{x}^*)$, $T(\bar{x})=S(\bar{x})$, and $T(\bar{x}^*)=S(\bar{x}^*)$, it follows that

$$U\big(S(\bar{x})\big)=S(\bar{x}) \text{ and } U\big(S(\bar{x}^*)\big)=S(\bar{x}^*).$$

Therefore, $S(\bar{x})$ and $S(\bar{x}^*)$ are fixed points of U on $S(E)$. Using the fact that $S(\bar{x})$ and $S(\bar{x}^*)$ are fixed points of U such that $\rho\left(U\big(S(\bar{x})\big)-U\big(S(\bar{x}^*)\big)\right)<\infty$, then according to Theorem 8, we have

$$S(\bar{x})=S(\bar{x}^*).$$

Since S is injective and $S(\bar{x})=S(\bar{x}^*)$, then we obtain $\bar{x}=\bar{x}^*$. In other words, the coincidence point of T and S is unique. Moreover, since T and S are weakly-compatible, we get

$$T\circ S(\bar{x})=S\circ T(\bar{x}).$$

Thus, we have

$$T\big(T(\bar{x})\big)=T\big(S(\bar{x})\big)=T\circ S(\bar{x})=S\circ T(\bar{x})=S\big(T(\bar{x})\big)=S\big(S(\bar{x})\big).$$

Therefore, $T(\bar{x})=S(\bar{x})$ is a coincidence point of T and S. Using the property of the uniqueness of coincidence point of T and S, it follows that $T(\bar{x})=S(\bar{x})=\bar{x}$. So, it is true that T and S have a unique common fixed point, that is $\bar{x}$. ■

The following theorem state the fixed point theorem for ρ-Suzuki-contraction mappings in modular spaces which is a generalization of the fixed point theorem in metric spaces with some extra assumptions.

Theorem 11. Let X_ρ be a modular space, $C \subseteq X_\rho$ be a ρ-compact set, $T: C \to C$ be a mapping, and ρ satisfies Δ_2-type condition, i.e. there exists $K > 0$ such that $\rho(2x) \leq K\rho(x)$, for all $x \in X_\rho$. If $K \leq 1$ and for all $x, y \in X$ such that $\frac{1}{2}\rho\big(x - T(x)\big) < \rho(x - y)$ implies

$\rho\big(T(x) - T(y)\big) < \rho(x - y)$,

then T has a unique fixed point.

Proof. Since $\rho\big(x - T(x)\big) \geq 0$, for all $x \in C$, then the set $\{\rho\big(x - T(x)\big) \,|\, x \in C\}$ is bounded bellow so that the infimum exists. Suppose

$$\beta = \inf\{\rho\big(x - T(x)\big) \mid x \in C\},$$

then for arbitrary $\varepsilon > 0$, there exists $x_\varepsilon \in C$ such that $\rho\big(x_\varepsilon - T(x_\varepsilon)\big) < \beta + \varepsilon$. Therefore, we get

1) for $\varepsilon = 1$, there exists $x_1 \in C$ such that $\rho\big(x_1 - T(x_1)\big) < \beta + 1$,
2) for $\varepsilon = \frac{1}{2}$, there exists $x_2 \in C$ such that $\rho\big(x_2 - T(x_2)\big) < \beta + \frac{1}{2}$, and
3) for $\varepsilon = \frac{1}{3}$, there exists $x_3 \in C$ such that $\rho\big(x_3 - T(x_3)\big) < \beta + \frac{1}{3}$.

Continue in this way to obtain a sequence $\{x_n\} \subseteq C$ such that

$$\rho\big(x_n - T(x_n)\big) < \beta + \frac{1}{n}, \text{ for all } n \in \mathbb{N}.$$

Moreover, it follows that

$$\beta - \frac{1}{n} < \beta \leq \rho\big(x_n - T(x_n)\big) < \beta + \frac{1}{n}, \text{ for all } n \in \mathbb{N}.$$

Therefore, there exists a sequence $\{x_n\} \subseteq C$ such that $\big|\rho\big(x_n - T(x_n)\big) - \beta\big| < \frac{1}{n}$, for all $n \in \mathbb{N}$.

Let $\mu > 0$ be given. According to the Archimedian property, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{\mu} < n_0$ or $\frac{1}{n_0} < \mu$. Thus, for all $n \in \mathbb{N}$ such that $n \geq n_0$, we have

$$\big|\rho\big(x_n - T(x_n)\big) - \beta\big| < \frac{1}{n} \leq \frac{1}{n_0} < \mu.$$

So, we have $\lim_{n\to\infty} \rho\big(x_n - T(x_n)\big) = \beta$. Since C is ρ-compact, without loss of generality, we may assume that there exist $v, w \in C$ such that $\{x_n\}$ and $\{T(x_n)\}$ ρ-convergent to v and w respectively. Next, we will show that $\beta = 0$. We may assume that $\beta > 0$. Let $\mu > 0$ be given. Since $\{x_n\}$ and $\{T(x_n)\}$ ρ-convergent to v and w respectively, there exists $n_1, n_2 \in \mathbb{N}$ such that

1) $\rho(x_n - v) < \frac{\mu}{2}$, for all $n \geq n_1$, and
2) $\rho(T(x_n) - w) < \frac{\mu}{2}$, for all $n \geq n_2$.

Using the assumption that ρ satisfies Δ_2-type condition such that $K \leq 1$, we obtain

$$\rho(x-y) = \rho\left(\frac{1}{2}2(x-z) + \frac{1}{2}2(z-y)\right)$$
$$\leq \rho\big(2(x-z)\big) + \rho\big(2(z-y)\big)$$
$$\leq K\,\rho(x-z) + K\rho(z-y)$$
$$\leq \rho(x-z) + \rho(z-y),$$

for all $x, y, z \in X_\rho$. Note that for all $n \in \mathbb{N}$, we have

1) $\rho(x_n - w) \leq \rho(x_n - v) + \rho(v - w)$ so that
$$\rho(x_n - w) - \rho(v - w) \leq \rho(x_n - v) \text{ and}$$
2) $\rho(v - w) \leq \rho(v - x_n) + \rho(x_n - w)$ so that
$$\rho(v - w) - \rho(x_n - w) \leq \rho(v - x_n).$$

Thus, we get

$$|\rho(x_n - w) - \rho(v - w)| \leq \rho(x_n - v), \text{ for all } n \in \mathbb{N}.$$

Therefore, for all $n \geq n_1$, it follows that

$$|\rho(x_n - w) - \rho(v - w)| \leq \rho(x_n - v) < \frac{\mu}{2} < \mu.$$

So, we have $\lim_{n\to\infty} \rho(x_n - w) = \rho(v - w)$. For all $n \in \mathbb{N}$, we have

1) $\rho\big(x_n - T(x_n)\big) \leq \rho(x_n - v) + \rho\big((v - w)\big) + \rho\big(w - T(x_n)\big)$ so that
$$\rho\big(x_n - T(x_n)\big) - \rho(v - w) \leq \rho(x_n - v) + \rho\big(w - T(x_n)\big)$$
2) $\rho(v - w) \leq \rho(v - x_n) + \rho\big(x_n - T(x_n)\big) + \rho(T(x_n) - w)$ so that
$$\rho(v - w) - \rho\big(x_n - T(x_n)\big) \leq \rho(v - x_n) + \rho(T(x_n) - w).$$

Thus, we get

$$\big|\rho\big(x_n - T(x_n)\big) - \rho(v - w)\big| < \rho(x_n - v) + \rho(T(x_n) - w), \text{ for all } n \in \mathbb{N}.$$

Let $n_3 = \max\{n_1, n_2\}$, then for all $n \geq n_3$, we have

$$\big|\rho\big(x_n - T(x_n)\big) - \rho(v - w)\big| < \rho(x_n - v) + \rho(T(x_n) - w) < \frac{\mu}{2} + \frac{\mu}{2} = \mu.$$

So, it follows that $\lim_{n\to\infty} \rho\big(x_n - T(x_n)\big) = \rho(v - w)$. Therefore, we obtain

$$\lim_{n\to\infty} \rho(x_n - w) = \rho(v - w) = \lim_{n\to\infty} \rho\big(x_n - T(x_n)\big) = \beta.$$

Furthermore, since $\lim_{n\to\infty}\rho(x_n - w) = \beta$, then for $\mu = \frac{1}{3}\beta$, there exists $n_4 \in \mathbb{N}$ such that for all $n \geq n_4$, we get

$\beta - \mu < \rho(x_n - w) < \beta + \mu.$

So that $\frac{2}{3}\beta < \rho(x_n - w)$. And, since $\lim_{n\to\infty}\rho\big(x_n - T(x_n)\big) = \beta$, then for $\mu = \frac{1}{3}\beta$, there existst $n_5 \in \mathbb{N}$ such that for all $n \geq n_5$, we get

$\beta - \mu < \rho\big(x_n - T(x_n)\big) < \beta + \mu.$

So that $\rho\big(x_n - T(x_n)\big) < \frac{4}{3}\beta$. Hence if $n_6 = \max\{n_4, n_5\}$, it follows that for all $n \geq n_6$, we have

$$\frac{2}{3}\beta < \rho(x_n - w) \text{ dan } \rho\big(x_n - T(x_n)\big) < \frac{4}{3}\beta.$$

Therefore, for all $n \geq n_6$, we obtain

$\frac{1}{2}\rho\big(x_n - T(x_n)\big) < \rho(x_n - w)$

that implies for all $n \geq n_6$, we get

$\rho\big(T(x_n) - T(w)\big) < \rho(x_n - w).$

Next, we will proof that $\rho\big(w - T(w)\big) = \lim_{n\to\infty}\rho\big(T(x_n) - T(w)\big)$. It is clear that

$$\left|\rho\big(T(x_n) - T(w)\big) - \rho\big(w - T(w)\big)\right| \leq \rho(T(x_n) - w), \text{ for all } n \in \mathbb{N}.$$

Thus, for all $n \geq n_2$, we get

$\left|\rho\big(T(x_n) - T(w)\big) - \rho\big(w - T(w)\big)\right| \leq \rho(T(x_n) - w) \leq \frac{\mu}{2} < \mu.$

So, we have $\rho\big(w - T(w)\big) = \lim_{n\to\infty}\rho\big(T(x_n) - T(w)\big)$. It follows that

$\rho\big(w - T(w)\big) = \lim_{n\to\infty}\rho\big(T(x_n) - T(w)\big) \leq \lim_{n\to\infty}\rho(x_n - w) = \beta.$

According to the definition of β, we obtain $\rho\big(w - T(w)\big) = \beta$ such that $\beta > 0$. Since

$\frac{1}{2}\rho\big(w - T(w)\big) < \rho\big(w - T(w)\big),$

then we have

$\rho\big(T(w) - T^2(w)\big) < \rho\big(w - T(w)\big) = \beta.$

Note that $T(C) \subseteq C$. Hence $T(w) \in C$. According to the definition of β, we get

$\rho\big(T(w) - T^2(w)\big) \geq \beta,$

which is a contradiction. So, it is true that $\beta = 0$. Next, we will proof that T has a fixed point. We may assume T does not has fixed points, then we get $T(x) \neq x$, for all $x \in C$ so that

$$\rho(x - T(x)) > 0, \text{ for all } x \in C.$$

Since $0 < \frac{1}{2}\rho(x_n - T(x_n)) < \rho(x_n - T(x_n))$, we have

$$\rho(T(x_n) - T^2(x_n)) < \rho(x_n - T(x_n)), \text{ for all } n \in \mathbb{N}.$$

Furthermore, we proof that $\lim_{n\to\infty} \rho(v, T(x_n)) = \rho(v, w)$. It is clear that

$$|\rho(v - T(x_n)) - \rho(v - w)| \leq \rho(T(x_n) - w), \text{ for all } n \in \mathbb{N}.$$

Thus, for all $n \geq n_2$, we get

$$|\rho(v - T(x_n)) - \rho(v - w)| \leq \rho(T(x_n) - w) < \frac{\mu}{2} < \mu.$$

So, we have $\lim_{n\to\infty} \rho(v - T(x_n)) = \rho(v - w)$ so that

$$\lim_{n\to\infty} \rho(v - T(x_n)) = \rho(v - w) = \lim_{n\to\infty} \rho(x_n - T(x_n)) = \beta = 0$$

that implies $\{T(x_n)\}$ ρ-convergent to v. Since

$$\rho(v - T^2(x_n)) \leq \rho(v - T(x_n)) + \rho(T(x_n) - T^2(x_n)), \text{ for all } n \in \mathbb{N},$$

it follows that

$$\lim_{n\to\infty} \rho(v - T^2(x_n)) \leq \lim_{n\to\infty}\{\rho(v - T(x_n)) + \rho(T(x_n) - T^2(x_n))\} = 0.$$

In other words, the sequence $\{T^2(x_n)\}$ ρ-convergent to v. Next, we may assume there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2}\rho(x_n - T(x_n)) \geq \rho(x_n - v) \text{ dan } \frac{1}{2}\rho(T(x_n) - T^2(x_n)) \geq \rho(T(x_n) - v).$$

So, we have

$$\begin{aligned}\rho(x_n - T(x_n)) &\leq \rho(x_n - v) + \rho(T(x_n) - v)\\ &\leq \frac{1}{2}\rho(x_n - T(x_n)) + \frac{1}{2}\rho(T(x_n) - T^2(x_n))\\ &< \frac{1}{2}\rho(x_n - T(x_n)) + \frac{1}{2}\rho(x_n - T(x_n)) = \rho(x_n - T(x_n)),\end{aligned}$$

which is a contradiction. Hence for all $n \in \mathbb{N}$, either

$$\frac{1}{2}\rho(x_n - T(x_n)) < \rho(x_n - v) \text{ or } \frac{1}{2}\rho(T(x_n) - T^2(x_n)) < \rho(T(x_n) - v)$$

hold so that either

$$\rho(T(x_n) - T(v)) < \rho(x_n - v) \text{ or } \rho(T^2(x_n) - T(v)) < \rho(T(x_n) - v)$$

holds. Hence one of the following holds:

1) There exists an infinite subset $I \subseteq \mathbb{N}$ such that $\rho(T(x_n) - T(v)) < \rho(x_n - v)$, for all $n \in I$.

2) There exists an infinite subset $J \subseteq \mathbb{N}$ such that $\rho\left(T^2(x_n) - T(v)\right) < \rho(T(x_n) - v)$, for all $n \in J$.

In the first case, we obtain

$$\rho\left(v - T(v)\right) = \lim_{n \in I, n \to \infty} \rho\left(T(x_n) - T(v)\right) < \lim_{n \in I, n \to \infty} \rho(x_n - v) = 0,$$

which implies $T(v) = v$. Also, in the second case, we obtain

$$\rho\left(v - T(v)\right) = \lim_{n \in J, n \to \infty} \rho\left(T^2(x_n) - T(v)\right) < \lim_{n \in I, n \to \infty} \rho(T(x_n) - v) = 0.$$

Hence, we have shown that v is a fixed point of T in both cases. This is a contradiction. Therefore, there exists $z \in C$ such that $T(z) = z$. Next, we will show that T has a unique fixed point. Let $y \in C$ be any fixed point of T, we will show that $y = z$. We may assume $y \neq z$. Since

$$\frac{1}{2}\rho\left(z - T(z)\right) = 0 < \rho(z - y),$$

we have

$$\rho\left(z - T(y)\right) = \rho\left(T(z) - T(y)\right) < \rho(z - y).$$

Hence, y is not a fixed point of T, which is a contradiction. So, it is true that $y = z$. Therefore, the fixed point z of T is unique. ■

The following theorem state the coincidence point theorem for ρ-Suzuki-contraction mapping in modular spaces.

Theorem 12. Let X_ρ be a modular space, $C \subseteq X_\rho$ be a nonempty set, $T, S: C \to C$ be such that $T(C) \subseteq S(C)$ and $S(C) \subseteq C$ be a ρ-compact set. Assume that the modular ρ satisfies Δ_2-type condition, i.e. there exists $K > 0$ such that $\rho\left(2S(x)\right) \leq K\rho\left(S(x)\right)$, for all $x \in C$. If $K \leq 1$ and $\frac{1}{2}\rho\left(S(x) - T(x)\right) < \rho\left(S(x) - S(y)\right)$ implies

$$\rho\left(T(x) - T(y)\right) < \rho\left(S(x) - S(y)\right),$$

for all $x, y \in C$, then T and S have a coincidence point. Moreover, if T and S are weakly-compatible, then T and S have a unique common fixed point.

Proof. By Theorem 9, there exists $E \subseteq C$ such that $S(E) = S(C)$ and $S: E \to C$ is injective. Now, we define a map $h: S(E) \to S(E)$ by

$$h\left(S(x)\right) = T(x), \text{ for all } S(x) \in S(E).$$

Since S is injective, then analogous to the proof that U is a mapping on $S(E)$ in Theorem 10, it can be proven that h is a mapping on $S(E)$. Note that for all $S(x), S(y) \in S(E)$ such that $\frac{1}{2}\rho\left(S(x) - T(x)\right) < \rho\left(S(x) - S(y)\right)$ implies

$$\rho\big(T(x) - T(y)\big) < \rho(S(x) - S(y)).$$

Then, by using the definition of h, we have

$\rho\left(h(S(x)) - h(S(y))\right) < \rho(S(x) - S(y)).$

Since $S(E) = S(C)$ is ρ-compact, the modular ρ satisfies Δ_2-type condition such that $K \leq 1$, and for all $S(x), S(y) \in S(E)$ such that $\frac{1}{2}\rho\left(S(x) - h(S(x))\right) < \rho\big(S(x) - S(y)\big)$ implies

$$\rho\left(h(S(x)) - h(S(y))\right) < \rho(S(x) - S(y)),$$

then according to Theorem 11, we get h has a fixed point on $S(E)$. Thus, there exists $S(\bar{x}) \in S(E)$ such that $h\big(S(\bar{x})\big) = S(\bar{x})$ or $T(\bar{x}) = S(\bar{x})$. So, $\bar{x}$ is a common fixed point of T and S. Moreover, if T and S are weakly-compatible, we will show that T and S have a unique common fixed point. Let $\bar{x}^*$ be any coincidence point of T and S. Then we have

$$T(\bar{x}) = S(\bar{x}) \text{ dan } T(\bar{x}^*) = S(\bar{x}^*).$$

Since $h\big(S(\bar{x})\big) = T(\bar{x}), h\big(S(\bar{x}^*)\big) = T(\bar{x}^*), T(\bar{x}) = S(\bar{x})$, and $T(\bar{x}^*) = S(\bar{x}^*)$, we obtain

$$h\big(S(\bar{x})\big) = S(\bar{x}) \text{ and } h\big(S(\bar{x}^*)\big) = S(\bar{x}^*).$$

So, $S(\bar{x})$ and $S(\bar{x}^*)$ are fixed points of h on $S(E)$. Then, since $S(E) = S(C)$ is ρ-compact and for all $S(x), S(y) \in S(E)$ such that $\frac{1}{2}\rho\left(S(x) - h(S(x))\right) < \rho\big(S(x) - S(y)\big)$ implies

$\rho\left(h(S(x)) - h(S(y))\right) < \rho\big(S(x) - S(y)\big),$

then according to Theorem 11, we get h has a unique fixed point on $S(E)$ so that

$S(\bar{x}) = S(\bar{x}^*).$

Since S is injective and $S(\bar{x}) = S(\bar{x}^*)$, it follows that $\bar{x} = \bar{x}^*$. So, it is true that T and S have a unique common coincidence point, that is $\bar{x}$. Moreover, if T and S are weakly-compatible, then we have

$T \circ S(\bar{x}) = S \circ T(\bar{x})$

so that

$T\big(T(\bar{x})\big) = T\big(S(\bar{x})\big) = T \circ S(\bar{x}) = S \circ T(\bar{x}) = S\big(T(\bar{x})\big) = S\big(S(\bar{x})\big).$

Thus, $T(\bar{x}) = S(\bar{x})$ is also a coincidence point of T and S. By the uniqueness of a coincidence point of T and S, we get $T(\bar{x}) = S(\bar{x}) = \bar{x}$. So, $\bar{x}$ is a common fixed point of T and S. Therefore, it is true that T and S have a unique common fixed point, that is $\bar{x}$. ■

CONCLUSION

In this article, we give a quasi-contraction mapping in the modular space X_ρ which is called ρ-quasi-contraction mappings, i.e. an itself mapping T on a subset $C \subseteq X_\rho$ that is nonempty satisfies a condition which states there exists $k \in [0,1)$ such that for all $x, y \in C$ one has $\rho\big(T(x) - T(y)\big)$ is less than or equal to $k \max\{\rho(x-y), \rho\big(x - T(x)\big), \rho\big(y - T(y)\big), \rho\big(x - T(y)\big), \rho\big(y - T(x)\big)\}$ and we also give a Suzuki-contraction mapping on the modular space which is called ρ-Suzuki-contraction mappings, i.e. an itself mapping $T: C \to C$ that satisfies a condition which states for all $x, y \in C$ such that $\frac{1}{2}\,\rho\big(x - T(x)\big)$ is less than $\rho(x-y)$ implies $\rho\big(T(x) - T(y)\big)$ is less than $\rho(x-y)$. Moreover, we give the fixed point theorem for Suzuki-contraction mapping on modular spaces which is a generalization of the fixed point theorem for Suzuki-contraction mapping on metric spaces, but with the additional assumption that the modular ρ on the modular space X_ρ satisfies Δ_2-type condition and the constant that satisfies Δ_2-type condition is less than one. Furthermore, the coincidence point theorem for quasi-contraction mappings and Suzuki-contraction mappings on modular spaces is a generalization of the fixed point theorem by providing the conditions for two mappings T and S on the coincidence point theorem, one of the mappings, that is T, satisfies a condition such that the range of T is a subset of the range of the other mapping, that is S.

There are still many types of spaces which are more general than metric spaces and modular spaces. Therefore, it is possible that quasi-contraction mapping and Suzuki-contraction mapping can be defined in other spaces and the fixed point theorem and the coincidence point theorem can be developed.

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