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IDEALS IN MATRIX RINGS OVER COMMUTATIVE RINGS

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ABSTRACT

In this research, we discuss about ideal of matrix rings over commutative rings and its properties. The research of ideal in matrix rings is important because it is the basic structure for constructing factor rings in matrix rings. This research is a literature research that examines and develops research that has been done previously. We develop ideal concepts in an usually ring into matrix rings over commutative rings. By showing the sufficient and necessary condition of ideal of matrix rings over commutative rings, we show the form of ideal in matrix rings over commutative rings. Then, by using the properties of two ideal in a ring, we show the properties of intersection, addition and multiplication of two ideal in matrix rings over commutative rings. The result of this research is the form of ideal in matrix rings over commutative rings is the set of all matrices over the ideal in commutative rings. Then, the intersection, addition and multiplication of two ideal in matrix rings over commutative rings is also an ideal of matrix rings over commutative rings.
Keywords: Matrix Ring, Ideal, Commutative Ring, Properties of Two Ideal

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PRELIMINARY

In group theory, a new group can be formed by using a normal subgroup, and it is called the factor group. In ring theory, in line with the idea of forming factor groups, the factor rings can be formed. In the process of forming this factors ring, it motivates the emergence of an ideal definition of a ring. In ring theory, an ideal of a ring is a special subset of its elements. Let R is a ring, a subset I is called a left ideal of R if it is an additive subgroup of R that absorbs multiplication from the left by elements of R, that is, I is a left ideal if I is a subgroup of R under addition and for all $r \in R$ and $x \in I$, the product rx is in I. The discussion about properties of two ideal of ring has been studied, i.e the intersection, addition, and product of two ideal in a commutative ring are also ideal (Salayan & Siregar, 2020) and flat-ideal in integral domain in (Kim & Lim, 2020).

Matrix is one material that is quite familiar in the discussion of mathematics. In linear algebra, the discussion of matrices is carried out on real or complex fields (Boyd & Vandenberghe, 2018), (Chasnov, 2018). Furthermore, the discussion of this matrix can be extended to the ring objects. In mathematics, rings are the algebraic structures that generalize fields where multiplication do not have to be commutative and multiplicative inverses do not have to exist. In other words, a ring is a set equipped with two binary operations satisfying properties analogous to those of addition and multiplication of intergers. Ring elements may be numbers such as integers or complex numbers, but they may also be non-numerical objects such as polynomials, square matrices, functions, and power series. Then, the discussion about the extension of ring in fuzzy set has been studied in (Pratama, 2022).

The discussion about the matrix over the ring has been done. The discussion about matrix product codes over commutativer rings has been studied in (Boulagouaz & Deajim, 2021). Meanwhile, Furthermore, the discussion about rank of matrix over the commutative ring is studied by (Ismanto, 2018). Several other studies related to the ring matrix, i.e the zero divisors in matrix over the commutativer ring (Nurhayani et al., 2019). Furthermore, the study about clean matrix over ring of power series has been discussed in (Rugayah et al., 2021) and (Faisol & Fitriani, 2021a).

Suppose R is a commutative ring and $M_{n\times n}(R)$ is the set of all $n \times n$ matrices over R. The addition and multiplication in $M_{n\times n}(R)$ are defined as addition and multiplication of matrix as in linear algebra. The algebraic stucture of $M_{n\times n}(R)$ is a ring with unity under matrix addition and matrix multiplication, and $M_{n\times n}(R)$ is called matrix ring. The development research about matrix ring has been done. The discussion about characterization of matrix rings has discussed by (Petik, 2021) and (Goyal & Khurana, 2023). The development research about the exixtence of idompotent and clean elemen in matrix rings has been discussed in (Ambarsari et al., 2019). The discussion about matrix rings related to triangular matrices has been discussed among others in (Bennis et al., 2021). Furthermore, the discussion about matrix rings related to homomorphisme has been discussed among others in (Krylov & Tuganbaev, 2023) and (Faisol & Fitriani, 2021b).

In this article, we will discuss about ideal of matrix ring $M_{n\times n}(R)$. We will discuss about the sufficient and necessary condition of ideal in matrix ring $M_{n\times n}(R)$. Furthermore, it also will discussed about the properties of ideal in matrix ring $M_{n\times n}(R)$. Since matrix ring $M_{n\times n}(R)$ is not a commutative ring, then we will investigate enforceability the properties of intersection, addition, and product of two ideal in matrix ring $M_{n\times n}(R)$ as in (Salayan & Siregar, 2020). The ideal that is obtained of this research motivates the emergence potentially to constuct the factor ring in matrix ring $M_{n\times n}(R)$.

This section is the prelimanary topics in this research which contains subring and ideal with their properties in an usually rings. In a ring theory, an ideal of a ring is a special subset of its elements. As in integers, ideals generalize certain subsets of the integers, such as the even numbers or the multiples of an integers. The following definition explains about subring and its property in an usually rings.

Definition 1. (Judson, 2022) *Let R be a ring and S is a non-empty subset of R, then S is called a subring of R if S is also a ring under the same binary operations of R.*

The following property shows the conditions for a non-empty subset of a ring is a subring.

Lemma 2. (Judson, 2022) *Let R be a ring and S is a non-empty subset of R. The subset S of R is a subring of R if and only if $x - y \in S$ and $xy \in S$ for all $x, y \in S$.*

An ideal is a subring of a ring that has special properties. The following is given the definition of an ideal in a rings.

Definition 3. (Judson, 2022) *Let R be a ring and I is a non-empty subset of R, then I is called an ideal of R if I subring of R and $ar \in I$ and $ra \in I$, for all $a \in I, r \in R$.*

The following property discusses about the relation of two ideal in a special condition of R, i.e for R *is* a commutative ring.

Theorem 4. (Salayan & Siregar, 2020) *Let R be a commutative ring and I_1, I_2 is an ideal of R, respectively, then* :

1. $I_1 \cap I_2$ *is an ideal of* R
2. $I_1 + I_2$ *is an ideal of* R
3. $I_1 I_2$ *is an ideal of* R.

METHODS

This research is a literature research that examines and develops research that has been done previously, i.e ideal of matrix ring $M_{n\times n}(R)$ and investigation the properties of two ideal of matrix ring $M_{n\times n}(R)$ as in (Salayan & Siregar, 2020). The steps taken in this research are as follows :

1. Literature review about the ideal of a ring and matrix ring;
2. Show that I is an ideal of R is a sufficient condition for $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$;
3. Show that I is an ideal of R is a necessary condition for $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$;

4. Show that $M_{n\times n}(I_1 \cap I_2)$ and $M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$, respectively, and $M_{n\times n}(I_1 \cap I_2) = M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$,
5. Show that $M_{n\times n}(I_1 + I_2)$ and $M_{n\times n}(I_1) + M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$, respectively, and $M_{n\times n}(I_1 + I_2) = M_{n\times n}(I_1) + M_{n\times n}(I_2)$,
6. Show that $M_{n\times n}(I_1 I_2)$ and $M_{n\times n}(I_1)M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$, respectively, and $M_{n\times n}(I_1 I_2) = M_{n\times n}(I_1)M_{n\times n}(I_2)$.

The following figure is the flowchart in this research.

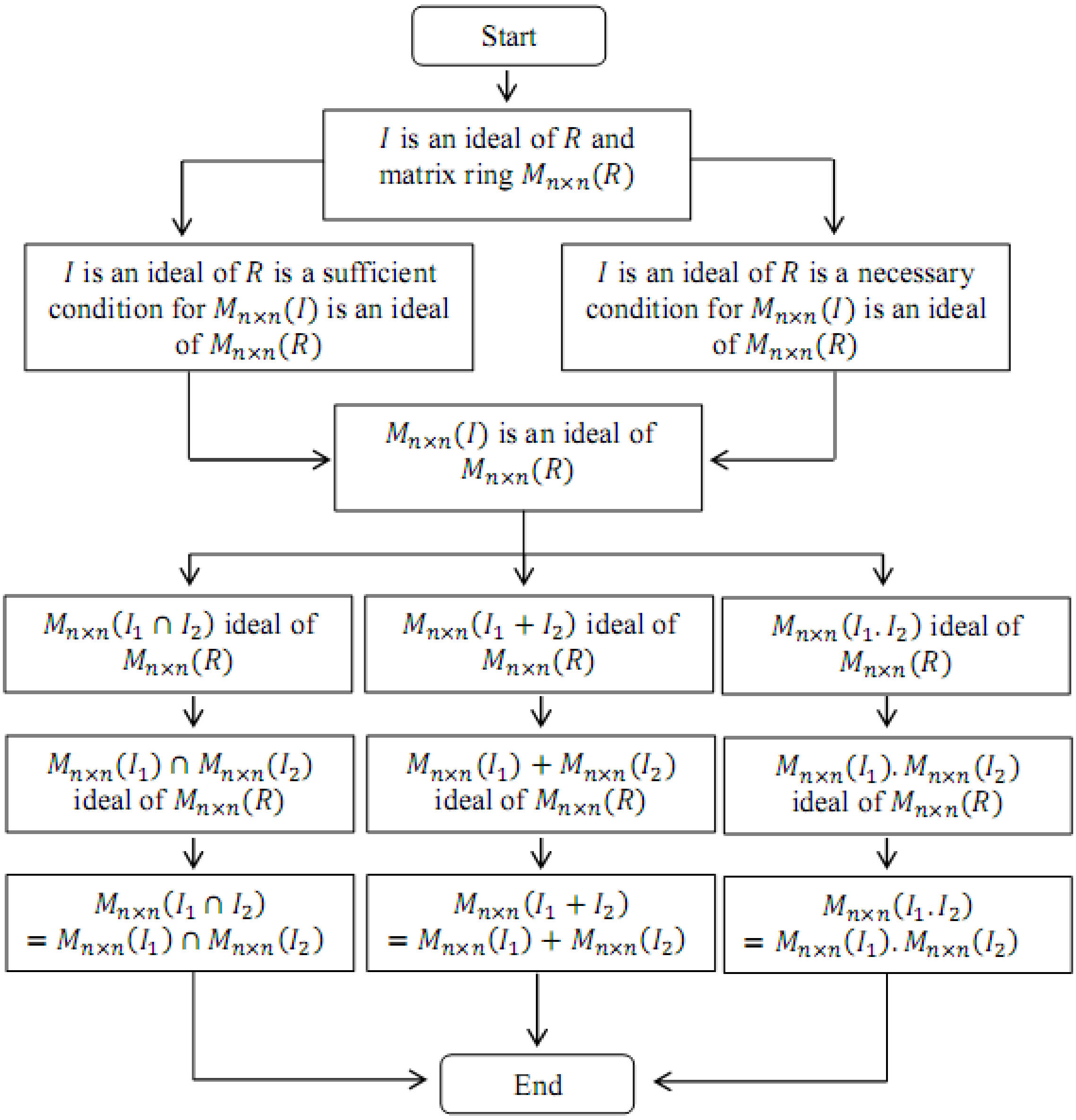

Figure 1. Flowchart of Research

RESULT AND DISCUSSION

This section is the main part of this research which contains sufficient and necessary conditions for ideal in matrix rings over commutative rings. Further more, it also contains some properties of ideal in matrix rings over commutative rings. Let $M_{n\times n}(R)$ is matrix rings over commutative rings R. The following lemma shows that $M_{n\times n}(I)$ is subring of $M_{n\times n}(R)$, where I is an ideal of R.

Lemma 5. *Let R is a commutative ring. If I is an ideal of R, then*

$$M_{n\times n}(I) = \{A|[A]_{ij} \in I, \text{for } i,j = 1,2,\dots,n\}$$

is a subring of $M_{n\times n}(R)$.

Proof. Since I is an ideal of R, then I is a subring of R. Consequently, for all $a,b \in I$, $a - b \in I$ and $ab \in I$. It will shown that $M_{n\times n}(I)$ is a subring of $M_{n\times n}(R)$. Let any $A = [a_{ij}]$, $B = [b_{ij}] \in M_{n\times n}(I)$, where $a_{ij}, b_{ij} \in I$ and $i,j = 1,2,\dots n$. Note that

$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$$

$$AB = [a_{ij}][b_{ij}] = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right].$$

Since $a_{ij}, b_{ij} \in I$ then $a_{ij} - b_{ij} \in I$ and $\sum_{k=1}^{n} a_{ik}b_{kj} \in I$. Therefore, $[a_{ij} - b_{ij}] \in M_{n\times n}(I)$ and consequently $A - B \in M_{n\times n}(I)$. Furthermore, $[\sum_{k=1}^{n} a_{ik}b_{kj}] \in M_{n\times n}(I)$ and consequently $AB \in M_{n\times n}(I)$. So, $M_{n\times n}(I)$ is a subring of $M_{n\times n}(R)$. ■

According to Lemma 5, it is obtained the following theorem which shows the sufficient condition of ideal in matrix rings over commutative rings R.

Theorem 6. *Let R is a commutative ring. If I is an ideal of R, then*

$$M_{n\times n}(I) = \{A|[A]_{ij} \in I, \text{for } i,j = 1,2,\dots,n\}$$

is an ideal of $M_{n\times n}(R)$.

Proof. According to Lemma 5, $M_{n\times n}(I)$ is a subring of $M_{n\times n}(R)$. Note that, for all $A = [a_{ij}] \in \mathrm{M}_{n\times n}(I)$ and $S = [s_{ij}] \in \mathrm{M}_{n\times n}(R)$ where $a_{ij} \in I$ and $s_{ij} \in R$, $i,j = 1,2,\dots,n$ then

$$AS = [a_{ij}][s_{ij}] = \left[\sum_{k=1}^{n} a_{ik}s_{kj}\right]$$

$$AS = [s_{ij}][a_{ij}] = \left[\sum_{k=1}^{n} s_{ik}a_{kj}\right].$$

Since $a_{ij} \in I$ and $s_{ij} \in R$ for $i,j = 1,2,\dots,n$, then $a_{ij}s_{ij} \in I$ and $s_{ij}a_{ij} \in I$. Furthermore, $\sum_{k=1}^{n} a_{ik}s_{kj} \in I$ and $\sum_{k=1}^{n} s_{ik}a_{kj} \in I$. Therefore, $[\sum_{k=1}^{n} a_{ik}s_{kj}] \in M_{n\times n}(I)$ and

$\left[\sum_{k=1}^{n} s_{ik}a_{kj}\right] \in M_{n\times n}(I)$. Consequently, $AS \in M_{n\times n}(I)$ and $SA \in M_{n\times n}(I)$. It is proved that $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$. ■

According to Theorem 6, it is obtained that I is an ideal of R is a sufficient condition for $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$. Then, the following example shows the form of ideal in matrix ring of order 2×2 over the set of all integers $\mathbb{Z}$.

Example 7. Let n is an integer and $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$. Then, $M_{2\times 2}(n\mathbb{Z})$ is a subring of $M_{2\times 2}(\mathbb{Z})$, and furthermore, $M_{2\times 2}(n\mathbb{Z})$ is an ideal of $M_{2\times 2}(\mathbb{Z})$. ■

The following properties in Lemma 8 and Theorem 9 show the neccessary condition of ideal in matrix rings over commutative rings R.

Lemma 8. *If* $M_{n\times n}(I) = \{\mathrm{A} \mid [\mathrm{A}]_{ij} \in I, \forall i,j = 1,2,\dots,n\}$ *is an ideal of* $M_{n\times n}(R)$, *then* I *is a subring of* R.

Proof. Since $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$, then $M_{n\times n}(I)$ is a subring of $M_{n\times n}(R)$. So, for all $A,B \in M_{n\times n}(I)$, $A-B \in M_{n\times n}(I)$ and $AB \in M_{n\times n}(I)$. It will shown that I is a subring of R. Suppose

$$\mathrm{E}_{kk} = \begin{array}{c c} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & \text{row-}k \\ \text{column-}k & \end{array}$$

For any $a,b \in I$, it is obtained that

$$\begin{aligned} a\mathrm{E}_{kk} - b\mathrm{E}_{kk} &= a\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} - b\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a-b & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \\ &= (a-b)\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \\ &= (a-b)\mathrm{E}_{kk}, \end{aligned}$$

and

$$
\begin{aligned}
(a\mathrm{E}_{kk})(b\mathrm{E}_{kk}) &= a\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} b \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & ab & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \\
&= ab\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \\
&= (ab)\mathrm{E}_{kk}.
\end{aligned}
$$

Since $a, b \in I$, then a is an entry of some $A \in M_{n\times n}(I)$ and b is also an entry of some $B \in M_{n\times n}(I)$. Since $A, B \in M_{n\times n}(I)$, then $E_{ki}AE_{jl} = aE_{kl} \in M_{n\times n}(I)$ dan $E_{pi}BE_{jq} = bE_{pq} \in M_{n\times n}(I)$ where $a = [A]_{ij}$ dan $b = [B]_{ij}$ for $k, l, p, q = 1,2, \dots, n$. In order to show that $a - b \in I$ and $ab \in I$, it is simply taken condisition $k = l = p = q$. Note that, for $k = l = p = q$, it is obtained $aE_{kk} - bE_{kk} = (a - b)E_{kk}$ and $(aE_{kk})(bE_{kk}) = (ab)E_{kk}$. Since $aE_{kk}, bE_{kk} \in M_{n\times n}(I)$, then $(a - b)E_{kk} \in M_{n\times n}(I)$ and $(ab)E_{kk} \in M_{n\times n}(I)$. Therefore, $a - b \in I$ dan $ab \in I$. It shows that I is a subring of R.■

Theorem 9. *If* $\mathrm{M}_{n\times n}(I) = \{A|[A]_{ij} \in I, \forall i, j = 1,2, \dots, n\}$ *is an ideal of* $M_{n\times n}(R)$, *then* I *an ideal of* R.

Proof. According to Lemma 8, I is a subring of R. Let any $r \in R$ and $a \in I$. Since $r \in R$ consequently $rI_n \in M_{n\times n}(R)$. Let $a = [A]_{ij}$, it will shown that $ra \in I$ and $ar \in I$. Since $A \in M_{n\times n}(I)$, consequently $E_{ki}AE_{jl} = aE_{kl} \in M_{n\times n}(I)$ for all $k, l = 1,2, \dots, n$. Because of $M_{n\times n}(I)$ is an ideal, then $(rI_n)(aE_{kl}) = (ra)E_{kl} \in M_{n\times n}(I)$ and $(aE_{kl})(rI_n) = (ar)E_{kl} \in M_{n\times n}(I)$. Furthermore, since $(ra)E_{kl} \in M_{n\times n}(I)$ and $(ar)E_{kl} \in M_{n\times n}(I)$, then $ra \in I$ dan $ar \in I$. Therefore, I is an ideal of R. ■

According to Theorem 9, it is obtained that I is an ideal of R is also a necessary condition for $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$. Furthermore, I is an ideal of R is an sufficient and necessary condition for $M_{n\times n}(I)$ is an ideal of $M_{n\times n}(R)$. Then, the form of ideal in matrix ring $M_{n\times n}(R)$ is $M_{n\times n}(I)$ for I is an ideal of R.

The following example shows the form of ideal in $\mathbb{Z}$, if it is known ideal of matrix ring of order $\mathbf{2 \times 2}$ over the set of all integers $\mathbb{Z}$.

Example 10. Let $M_{2\times 2}(n\mathbb{Z})$ is an ideal of $M_{2\times 2}(\mathbb{Z})$. Then, $n\mathbb{Z}$ for $n = 1,2,\ldots$ is a subring of $\mathbb{Z}$, and furthermore it is an ideal of $\mathbb{Z}$. ■

The following discussion studies about the properties of two ideals in matrix rings over commutative rings, i.e intersection of two ideals in matrix rings over commutative rings. In an usual ring, suppose I_1 and I_2 is an ideal of ring, respectively, then

$$I_1 \cap I_2 = \{x|x \in I_1 \text{ and } x \in I_2\}$$

is also an ideal of ring. The following properties discuss about applicability of ideal condition in the set of all matrix over $I_1 \cap I_2$.

Lemma 11. *If I_1 and I_2 is an ideal of R, respectively, then $M_{n\times n}(I_1 \cap I_2)$ is an ideal of $M_{n\times n}(R)$.*

Proof. Since I_1 and I_2 is an ideal of R, respectively, then $I_1 \cap I_2$ is also an ideal of R. Therefore, according to Theorem 6, it is obtained that $M_{n\times n}(I_1 \cap I_2)$ is an ideal of $M_{n\times n}(R)$. ■

Lemma 12. *If I_1 and I_2 is an ideal of R, respectively, then $M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$.*

Proof. Since I_1 and I_2 is an ideal of R, respectively, then according to Theorem 6, it is obtained that $M_{n\times n}(I_1)$ and $M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$, respectively. Consequently, intersection of them, $M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$. ■

According to Lemma 11 and Lemma 12, it is obtained that each of $M_{n\times n}(I_1 \cap I_2)$ and $M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$, for I_1 and I_2 is an ideal of R, respectively. Furthermore, it will be shown that intersection of two ideals in in matrix rings over commutative rings i.e the form $M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ is equal to $M_{n\times n}(I_1 \cap I_2)$.

Theorem 13. *If I_1 and I_2 is an ideal of R, respectively, then*

$$M_{n\times n}(I_1 \cap I_2) = M_{n\times n}(I_1) \cap M_{n\times n}(I_2).$$

Proof. In order to show $M_{n\times n}(I_1 \cap I_2) = M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$, it will shown that $M_{n\times n}(I_1 \cap I_2) \subseteq M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ and $M_{n\times n}(I_1) \cap M_{n\times n}(I_2) \subseteq M_{n\times n}(I_1 \cap I_2)$. Let any $A = [a_{ij}] \in M_{n\times n}(I_1 \cap I_2)$ for $i,j = 1,2,\ldots,n$, then $a_{ij} \in I_1 \cap I_2$. Since $a_{ij} \in I_1 \cap I_2$, then $a_{ij} \in I_1$ and $a_{ij} \in I_2$. Therefore, $A \in M_{n\times n}(I_1)$ and $A \in M_{n\times n}(I_2)$. Consequently $A \in M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$. So $M_{n\times n}(I_1 \cap I_2) \subseteq M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$.

Conversely, let any $A = [a_{ij}] \in M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ for $i,j = 1,2,\ldots,n$, then it is obtained $A \in M_{n\times n}(I_1)$ and $A \in M_{n\times n}(I_2)$. Therefore, $a_{ij} \in I_1$ and $a_{ij} \in I_2$. Furthermore, since $a_{ij} \in I_1$ and $a_{ij} \in I_2$, then $a_{ij} \in I_1 \cap I_2$ and therefore, $A \in M_{n\times n}(I_1 \cap I_2)$. Consequently, it is obtained $\mathrm{M}_{n\times n}(I_1) \cap \mathrm{M}_{n\times n}(I_2) \subseteq \mathrm{M}_{n\times n}(I_1 \cap I_2)$. ■

According to Theorem 13, matrix over intersection of two ideals in R is equal to intersection of two matrix over those ideals, i.e $M_{n\times n}(I_1 \cap I_2) = M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$. The following discussion studies about the properties of two ideals in matrix rings over commutative rings, i.e addition of two ideals in matrix rings over commutative rings. In an usual ring, suppose I_1 and I_2 is an ideal of ring, respectively, then

$$I_1 + I_2 = \{x + y | x \in I_1 \text{ and } y \in I_2\}$$

is also an ideal of ring. The following properties discuss about applicability of ideal condition in the set of all matrix over $I_1 + I_2$.

Lemma 14. *If I_1 and I_2 is an ideal of R, respectively, then $M_{n\times n}(I_1 + I_2)$ is an ideal of $M_{n\times n}(R)$.*

Proof. Since I_1 and I_2 is an ideal of R, respectively, then $I_1 + I_2$ is also an ideal of R. Therefore, $M_{n\times n}(I_1 + I_2)$ is an ideal of $M_{n\times n}(R)$. ■

Lemma 15. *If I_1 and I_2 is an ideal of R, respectively, then $M_{n\times n}(I_1) + M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$.*

Proof. Since I_1 and I_2 is an ideal of R, respectively, then each of $M_{n\times n}(I_1)$ and $M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$. Therefore, the set $M_{n\times n}(I_1) + M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$. ■

According to Lemma 14 and Lemma 15, it is obtained that each of $M_{n\times n}(I_1 + I_2)$ and $M_{n\times n}(I_1) + M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$, for I_1 and I_2 is an ideal of R, respectively. Furthermore, it will be shown that addition of two ideals in in matrix rings over commutative rings i.e $M_{n\times n}(I_1) + M_{n\times n}(I_2)$ is equal to $M_{n\times n}(I_1 + I_2)$.

Theorem 16. *If I_1 and I_2 is an ideal of R, respectively, then*

$$M_{n\times n}(I_1 + I_2) = M_{n\times n}(I_1) + M_{n\times n}(I_2).$$

Proof. It will be shown that $M_{n\times n}(I_1 + I_2) \subseteq M_{n\times n}(I_1) + M_{n\times n}(I_2)$ and $M_{n\times n}(I_1) + M_{n\times n}(I_2) \subseteq M_{n\times n}(I_1 + I_2)$. Let $A = [a_{ij}] \in M_{n\times n}(I_1 + I_2)$ for $i, j = 1,2, \dots, n$, then $a_{ij} \in I_1 + I_2$. Let $A = B + C = [b_{ij} + c_{ij}]$ where $b_{ij} + c_{ij} \in I_1 + I_2$ for all $i, j = 1,2, \dots, n$. Since $a_{ij} \in I_1 + I_2$, then $a_{ij} = b_{ij} + c_{ij}$ for $b_{ij} \in I_1$ and $c_{ij} \in I_2$. Therefore $B = [b_{ij}] \in M_{n\times n}(I_1)$ and $C = [c_{ij}] \in M_{n\times n}(I_2)$. Because of $A = B + C$, then $A \in M_{n\times n}(I_1) + M_{n\times n}(I_2)$. So, it is shown that $M_{n\times n}(I_1 + I_2) \subseteq M_{n\times n}(I_1) + M_{n\times n}(I_2)$.

Conversely, let $A = [a_{ij}] \in M_{n\times n}(I_1) + M_{n\times n}(I_2)$ for all $i, j = 1,2, \dots, n$. Suppose $A = B + C$ where $B = [b_{ij}] \in M_{n\times n}(I_1)$ and $C = [c_{ij}] \in M_{n\times n}(I_2)$. Therefore, $b_{ij} \in I_1$ and $c_{ij} \in I_2$. Consequently $b_{ij} + c_{ij} = a_{ij} \in I_1 + I_2$, and it is obtained $A \in M_{n\times n}(I_1 + I_2)$. So, $M_{n\times n}(I_1) + M_{n\times n}(I_2) \subseteq M_{n\times n}(I_1 + I_2)$. ■

According to Theorem 16, matrix over the sum of two ideals in R is equal to the sum of two matrix over those ideals, i.e $M_{n\times n}(I_1 + I_2) = M_{n\times n}(I_1) + M_{n\times n}(I_2)$. The following discussion studies about the properties of two ideals in matrix rings over commutative rings, i.e multiplication of two ideals in matrix rings over commutative rings. In an usual ring, suppose I_1 and I_2 is an ideal of ring, respectively, then

$$I_1 I_2 = \left\{ \sum_{i=1}^{n} p_i q_i \mid p_i \in I_1, q_i \in I_2, n = 1,2, \dots \right\}$$

is also an ideal of ring. The following properties discuss about applicability of ideal condition in the set of all matrix over $I_1 I_2$.

Lemma 17. *If I_1 and I_2 is an ideal of R, respectively, then $M_{n\times n}(I_1 I_2)$ is an ideal of $M_{n\times n}(R)$.*

Proof. According to the properties in an usual ring, if I_1 and I_2 is an ideal of R, respectively, then $I_1 I_2$ is also an ideal of R. Therefore, using Theorem 2, it is obtained $\mathrm{M}_{n\times n}(I_1 I_2)$ is an ideal of $\mathrm{M}_{n\times n}(R)$.■

Lemma 18. *If I_1 and I_2 an ideal of R, respectively, then $M_{n\times n}(I_1)M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$.*

Proof. Note that I_1 and I_2 is an ideal of R, respectively. Therefore, $M_{n\times n}(I_1)$ and $M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$, respectively. Consequently, $M_{n\times n}(I_1)M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$. ■

According to Lemma 17 and Lemma 18, it is obtained that each of $M_{n\times n}(I_1 I_2)$ and $M_{n\times n}(I_1).M_{n\times n}(I_2)$ is an ideal of $M_{n\times n}(R)$, for I_1 and I_2 is an ideal of R, respectively. Then, it will be shown that multiplication of two ideals in in matrix rings over commutative rings i.e $M_{n\times n}(I_1)M_{n\times n}(I_2)$ is equal to $M_{n\times n}(I_1 I_2)$.

Theorem 19. *If I_1 and I_2 is an ideal of R, respectively, then*

$$M_{n\times n}(I_1 I_2) = M_{n\times n}(I_1)M_{n\times n}(I_2)$$

Proof. It will be shown that $M_{n\times n}(I_1 I_2) \subseteq M_{n\times n}(I_1)M_{n\times n}(I_2)$ and $M_{n\times n}(I_1)M_{n\times n}(I_2) \subseteq M_{n\times n}(I_1 I_2)$. Let $A = [a_{ij}] \in M_{n\times n}(I_1 I_2)$ for $i, j = 1,2, \dots, n$, then $a_{ij} \in I_1 I_2$. Suppose $A = BC$ where $BC = \left[\sum_{k=1}^{n} b_{ik} c_{kj}\right]$ and $\sum_{k=1}^{n} b_{ik} c_{kj} \in I_1 I_2$ for all $i, j = 1,2, \dots, n$. Since $a_{ij} \in$

I_1I_2, then it can be written as $a_{ij} = \sum_{k=1}^{n} b_{ik}c_{kj}$ for all $b_{ij} \in I_1$ and $c_{ij} \in I_2$. Therefore, $B = [b_{ij}] \in M_{n\times n}(I_1)$ and $C = [c_{ij}] \in M_{n\times n}(I_2)$. Since $A = BC$ then matrix $A \in M_{n\times n}(I_1)M_{n\times n}(I_2)$. So, it is obtained that $M_{n\times n}(I_1I_2) \subseteq M_{n\times n}(I_1)M_{n\times n}(I_2)$.

Conversely, let $A = [a_{ij}] \in M_{n\times n}(I_1)M_{n\times n}(I_2)$ for $i,j = 1,2,\ldots,n$. Suppose $A = BC$ where $B = [b_{ij}] \in M_{n\times n}(I_1)$ and $C = [c_{ij}] \in M_{n\times n}(I_2)$. Therefore, $b_{ij} \in I_1$ and $c_{ij} \in I_2$. Note that $\sum_{k=1}^{n} b_{ik}c_{kj} = a_{ij} \in I_1I_2$, so then $A \in M_{n\times n}(I_1I_2)$. Consequently, $M_{n\times n}(I_1)M_{n\times n}(I_2) \subseteq M_{n\times n}(I_1I_2)$. Therefore, $M_{n\times n}(I_1I_2) = M_{n\times n}(I_1)M_{n\times n}(I_2)$.■

According to Theorem 19, matrix over the multiplication of two ideals in R is equal to the multiplication of two matrix over those ideals, i.e $M_{n\times n}(I_1I_2) = M_{n\times n}(I_1)M_{n\times n}(I_2)$. The following example discusses about the intersection, addition and multiplication of two ideals in $M_{n\times n}(\mathbb{Z})$.

Example 20. Suppose that the set of all integers $\mathbb{Z}$ is a commutative ring with unity. Therefore, $M_{2\times 2}(\mathbb{Z})$ is a matrix ring. Note that $2\mathbb{Z}$ and $5\mathbb{Z}$ is an ideal of $\mathbb{Z}$, respectively. Since $2\mathbb{Z}$ and $5\mathbb{Z}$ is an ideal of $\mathbb{Z}$, respectively, then $M_{2\times 2}(2\mathbb{Z})$ and $M_{2\times 2}(4\mathbb{Z})$ is also an ideal of $M_{2\times 2}(\mathbb{Z})$, respectively. Consequently, each of form $M_{2\times 2}(2\mathbb{Z}) \cap M_{2\times 2}(4\mathbb{Z})$, $M_{2\times 2}(2\mathbb{Z}) + M_{2\times 2}(4\mathbb{Z})$ and $M_{2\times 2}(2\mathbb{Z})M_{2\times 2}(4\mathbb{Z})$ is also an ideals of $M_{2\times 2}(\mathbb{Z})$, respectively. Furthermore, it is obtained that $M_{2\times 2}(2\mathbb{Z}) \cap M_{2\times 2}(4\mathbb{Z}) = M_{2\times 2}(2\mathbb{Z} \cap 4\mathbb{Z})$, $M_{2\times 2}(2\mathbb{Z}) + M_{2\times 2}(4\mathbb{Z}) = M_{2\times 2}(2\mathbb{Z} + 4\mathbb{Z})$ and $M_{2\times 2}(2\mathbb{Z})M_{2\times 2}(4\mathbb{Z}) = M_{2\times 2}(2\mathbb{Z}4\mathbb{Z})$. ■

CONCLUSION

Let R is a ring commutative and I is an ideal of R. Then, I is an ideal of R is a sufficient and also necessary condition for $M_{n\times n}(I)$ is an ideal of matrix ring $M_{n\times n}(R)$. So, the form ideal of matrix ring $M_{n\times n}(R)$ is $M_{n\times n}(I)$, for I is an ideal of R. Furthermore, if I_1 and I_2 is an ideal of R, then both of $M_{n\times n}(I_1 \cap I_2)$ and $M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$, and it is obtained $M_{n\times n}(I_1 \cap I_2) = M_{n\times n}(I_1) \cap M_{n\times n}(I_2)$. Moreover, each of $M_{n\times n}(I_1 + I_2)$ and $M_{n\times n}(I_1) + M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$, and it is obtained $M_{n\times n}(I_1 + I_2) = M_{n\times n}(I_1) + M_{n\times n}(I_2)$. Besides that, $M_{n\times n}(I_1I_2)$ and $M_{n\times n}(I_1)M_{n\times n}(I_2)$ is also an ideal of $M_{n\times n}(R)$, respectively, and it also be obtained $M_{n\times n}(I_1I_2) = M_{n\times n}(I_1)M_{n\times n}(I_2)$.

The future research can be done on investigation of the factor ring in matrix ring $M_{n\times n}(R)$. Ideal $M_{n\times n}(I)$ in $M_{n\times n}(R)$ can potentially be used to construct the factor ring in matrix ring $M_{n\times n}(R)$.

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