M A T H L I N E
JURNAL MATEMATIKA DAN PENDIDIKAN MATEMATIKA
ISSN 2502-5872 (Print)
ISSN 2622-3627 (Elektronik)

Volume 9 Number 4, November 2024, 1039-1049

THE IRREDUCIBLE UNITARY REPRESENTATION OF $SU(2)$ AND ITS LIE ALGEBRA REPRESENTATION

Edi Kurniadi[1*], Badrulfalah[2], Nurul Gusriani[3]
[1,2,3]Departement of Mathematics, Universitas Padjadjaran, West Java, Indonesia
*Correspondence: edi.kurniadi@unpad.ac.id

ABSTRACT

We study the three dimensional special unitary group $\mathrm{SU}(2) \subseteq \mathrm{GL}(n,\mathbb{C})$ whose the Lie algebra is given by $\mathfrak{su}(2) \subseteq \mathrm{M}(2,\mathbb{C})$. The research aims to construct a representation of $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$ realized on the inner product space $\mathbb{P}_{(1)}$ of all homogeneous polinomials of degree 2 and $\mathbb{P}_{(2)}$ of all homogeneous polinomials of degree 4 which satisfying irreducibility and unitarity conditions. Namely, The action of $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$ are presented on the spaces $\mathbb{P}_{(1)}$ and $\mathbb{P}_{(2)}$ respectively. In the first step, we computed all representations of $\mathrm{SU}(2)$ on $\mathbb{P}_{(1)}$ and $\mathbb{P}_{(2)}$. Furthermore, in the second step, by simply connectedness property of $\mathrm{SU}(2)$ then the irreducible unitary representation of Lie algebra $\mathfrak{su}(2)$ realized on $\mathbb{P}_{(2)}$ can be obtained from the $\mathrm{SU}(2)$ representation by using derived representation. The results showed the explicit formulas of representations of $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$
Keywords: Representation theory, $\mathrm{SU}(2)$, $\mathfrak{su}(2)$, Irreducible, Unitary, Homogeneous Polinomial.

How to Cite: Kurniadi, E., Badrulfalah, & Gusriani, N. (2024). The Irreducible Unitary Representation of $\mathrm{SU}(2)$ and Its Lie algebra Representations. *Mathline: Jurnal Matematika dan Pendidikan Matematika, 9*(4), 1039-1049. http://doi.org/10.31943/mathline.v9i4.653

PRELIMINARY

The research on representation theory of groups, Lie groups, and Lie algebras are vast, especially regarding irreducible unitary representations of matrix Lie groups and their Lie algebras (Bayard et al., 2017; Behzad et al., 2022; Eswara Rao, 2023; Feger et al., 2020; Hu et al., 2019; Liebeck et al., 2020; Nasr-Isfahani, 2015), and (Cowling, 2023; Hampsey et al., 2024) However, it is very necessary to have a model that is easy to understand in studying irreducible unitary representations of matrix Lie groups, particularly for young researchers. It is well known that $\mathrm{SU}(n)$ is one of matrix Lie group examples. In particular, for case $n = 2$ many researchers have studied this group and applied it to various fileds. These studies include open and dense subset of the special unitary matrix Lie group $\mathrm{SU}(2)$ where it can be a domain of new proper biharmonic functions (Gudmundsson, 2017) and operation of active $\mathrm{SU}(2)$ on poincare sphere (Saito, 2024). In addition, $\mathrm{SU}(2)$ can be applied to other fields. For examples, transformation of $\mathrm{SU}(2)$ can be applied to case of a general rogue wave solutions in the nonlinear Schrodinger equation (Pan et al., 2022) and progressive waves in Yang-Mills fields (Rabinowitch, 2024). Moreover, in the sense of other researches, researchers investigated it in the case of new BCH-like relation (Martínez-Tibaduiza et al., 2020) and Hamiltonian extended affine Lie algebra (Eswara Rao, 2023). It is interesting to study representation theory by using Mathematica (Feger et al., 2020) or another software to encourage mathematical problem solving(Susanto et al., 2023).

Furthermore, it is also well known that the Lie algebra of the special unitary matrix Lie group $\mathrm{SU}(2)$ is the Lie algebra $\mathfrak{su}(2) \subseteq \mathrm{M}(2,\mathbb{C})$ consisting of all 2×2 complex matrices with $A^* = -A$ and trace $(A) = 0$. In addition, using the simply connectedness of $\mathrm{SU}(2) \subseteq \mathrm{GL}(2,\mathbb{C})$, then all $\mathfrak{su}(2)$ representations with respect to its basis can be computed from representation of $\mathrm{SU}(2)$. In another research, $SU(2)$ can be related to meridian-traceless of $\mathrm{SU}(2)$-representations (Xie & Zhang, 2023). There are relations among matrix Lie algebras $\mathfrak{su}(1,1), \mathfrak{su}(2)$, and $\mathfrak{so}(2,1)$ (Martínez-Tibaduiza et al., 2020). Indeed, we also find some applications of Lie algebras in another field such as in deoxyribonucleic acid or DNA which consists of adenine, guanine, cytosine, and thymine (Fayazi et al., 2021; Hussain et al., 2024; Sánchez et al., 2006), algebraic topology for DNA (Braslavsky & Stavans, 2018), and in finance mathematics (Lim & Privault, 2016).

The motivation of this study comes from the question of Berndt (2007) corresponding to the representations of $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$. Different from previous result, in this present paper we present the explicit formulas for unitary irreducible representation of $\mathrm{SU}(2)$ and $\mathfrak{su}(2)$ on the space of two-complex homogeneous polynomials V_n of degree equals n. In our case, for $n = 1$ and $n = 2$. Furthermore, we give the explicit formula of a unitary irreducible representation of $\mathfrak{su}(2)$ computed from a unitary irreducible representation of $\mathrm{SU}(2)$ with respected the basis of $\mathfrak{su}(2)$.

METHODS

The steps used in solving problems in this research can be described as follows: Given the special unitary matrix Lie group $\mathrm{SU}(2)$. Then we realize the irreducible unitary representation (IUR) on two variables complex space V_n of degree n. In this case, the explicit formula of the IUR on V_n is obtained. Furthermore, since the matrix Lie group $\mathrm{SU}(2)$ is simply connected, then the associated IUR of the Lie algebra $\mathfrak{su}(2)$ can be computed as a derivation of the IUR of $\mathrm{SU}(2)$. Finally, we prove that the obtained representations of the matrix Lie group $\mathrm{SU}(2)$ and its Lie algebra $\mathfrak{su}(2)$ both are irreducible and unitary.

Moreover, in this section, we also provide some basic materials that we need to obtain the main research. We introduce the notion of the special Lie group $\mathrm{SU}(2)$ and its Lie algebra

$\mathfrak{su}(2)$, the notion of representation theory of a matrix Lie group, a unitary and irreducible representation of a matrix Lie groups. Let $\mathrm{GL}(n,\mathbb{C})$ be all $n \times n$ invertible matrices whose entries are complex numbers. In particular, for $n = 2$ we have the notion $\mathrm{GL}(2,\mathbb{C})$ of all 2×2 invertible matrices whose entries are complex numbers. Let $\mathrm{SU}(2)$ be a closed subgroup of $\mathrm{GL}(2,\mathbb{C})$ consisting of 2×2 invertible matrices satisfying $A^* = A^{-1}$ and its determinant be equals 1. Let $A \in \mathrm{SU}(2)$, then $A^* = A^{-1}$ and its determinant equals 1. Therefore, the necessary and sufficient condtions for the exponential matrix e^{sY} unitary is $(e^{sY})^* = (e^{sY})^{-1}$. In the other words, we have that $(e^{sY})^* = e^{-tY}$. But we also have that $(e^{sY})^* = e^{tY^*}$. Thus, $e^{-tY} = e^{tY^*}$. The latter equation holds for t iff $Y^* = -Y$. Furthermore, the condition of all matrices in $\mathrm{SU}(2)$ which have determinant 1 implies their traces in $\mathfrak{su}(2)$ equals 0. Formally, we state our arguments as follows:

Proposition 1. *Let* $\mathrm{SU}(2)$ *be a special Lie group consisting of all* 2×2 *matrices* A *with* $A^* = A^{-1}$ *and determinant be equals* 1. *Then the Lie algebra of* $\mathrm{SU}(2)$ *is given by* $\mathfrak{su}(2) \subseteq \mathrm{M}(2,\mathbb{C})$ *consisting of all* 2×2 *matrices* B *satisfying the conditions* $B^* = -B$ *and* $\mathrm{trace}(B) = 0$.

Definition 1. Let V be a vector space with finite dimension and let $G \subseteq \mathrm{GL}(\mathrm{n},\mathbb{C})$ be a matrix Lie group. A representation of G realized on the space V is a homomorphism of a Lie group G given by the map

$$\Sigma : G \to \mathrm{Aut}(V). \tag{1}$$

In the other words, Σ is homomorphism group as well as continuos. Furthermore, the representation Σ is called irreducible if V has no nontrivial invariant subspaces.

In addition, let V be an inner product space with $\dim V < \infty$. The representation (1) is called unitary if $\Sigma(g)$ is a unitary operator on V for all $g \in G$. In the other words,

$$\|\Sigma(g)v\| = \|v\| \tag{2}$$

for all $v \in V$.

Proposition 2. *Let G be a group and let π be a unitary matrix representation of G on* $\mathbb{C}^n$ *satisfying condition* $\pi(g) = S(g)$ *contained in* $\mathrm{U}(n)$. *In this case* $\mathrm{Aut}\,\mathbb{C}^n$ *is identified by* $\mathrm{GL}(n,\mathbb{C})$. *Then for every matrix* $N \in \mathrm{GL}(n,\mathbb{C})$ *satisfying* $NS(g) = S(g)N$, $N = zI_n, z \in \mathbb{C}$.

In the other words, if there is no exists such matrix then a finite unitary representation π *is irreducible.*

Definition 2. The representation $(\mathrm{d}\Sigma, \mathfrak{H}^{\infty})$ of $\mathfrak{g} = \mathrm{Lie}\ G$ of a continuous representation of a linear group $G - (\Sigma\ , \mathfrak{H})$ is given by

$$\mathrm{d}\Sigma(B)Y) \coloneqq \frac{d}{d\theta}\ \Sigma\left(\mathrm{e}^{\theta\mathrm{B}}\right)\mathrm{Y}_{\theta=0},\ \forall\, Y\ \in \mathfrak{H}^{\infty}. \tag{3}$$

RESULT AND DISCUSSION

Let $g = \begin{pmatrix} x_1 & x_2 \\ -\bar{x}_2 & \bar{x}_1 \end{pmatrix}$ be an element of $\mathrm{SU}(2)$ satisfying $\det g = 1$. Namely, we have $|X_1|^2 + |x_2|^2$ dan $\mathbb{P}_{(1)}$ be a space of complex homogeneous polynomials of degree 2 spanned by a basis S in the following form

$$H_{-1} = \frac{1}{\sqrt{2}} z_2^2,\quad H_0 = z_1 z_2,\quad H_1 = \frac{1}{\sqrt{2}} z_1^2 \tag{4}$$

Indeed, the dimension of $\mathbb{P}_{(1)}$ equals 3. Thus, for any $H \in \mathbb{P}_{(1)}$, it can be written in linear combinations of equation (4) as follows

$$H = \frac{1}{\sqrt{2}} k_0 z_2^2 + k_1 z_1 z_2 + \frac{1}{\sqrt{2}} k_2 z_1^2. \tag{5}$$

Our first result confirm result in (Berndt, 2007) and it is stated in the following propostion.

Proposition 3. *Let* $\Sigma_1(x)$ *be a linear transformation on the space* $\mathbb{P}_{(1)}$ *which is given by the explisit formula*

$$(\Sigma_1(g)H)(z_1, z_2) = H\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right), g \in \mathrm{SU}(2), \qquad \bar{z} = (z_1, z_2) \in \mathbb{C}^2. \tag{6}$$

Then the representation Σ_1 *of* $SU(2)$ *satisfies the irreducibility and unitarity conditions.*

Proof. The fisrst we shall prove the uniatry of Σ_1. Since $(\Sigma_1(g)H)(z_1, z_2) = H\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = H\left(\begin{pmatrix} \bar{x}_1 & -x_2 \\ \bar{x}_2 & x_1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = H(\bar{x}_1 z_1 - x_2 z_2, \bar{x}_2 z_1 + x_1 z_2) = \frac{1}{\sqrt{2}} k_0 (\bar{x}_1 z_1 - x_2 z_2)^2 + k_1(\bar{x}_1 z_1 - x_2 z_2)(\bar{x}_2 z_1 + x_1 z_2) + \frac{1}{\sqrt{2}} k_2(\ \bar{x}_2 z_1 + x_1 z_2)^2$, then is also homogoneous polynomial of degree 2. Moreover, to see that Σ_1 is a representation. Let $g_1, g_2 \in \mathrm{SU}(2)$ and $\bar{z} = (z_1, z_2) \in \mathbb{C}^2$. It can be computed as follows that

$$\begin{aligned}
(\Sigma_1(g_1 g_2)H)(z) &= H((g_1 g_2)^{-1}(z)) \\
&= H(g_2^{-1} g_1^{-1} z) = (\Sigma_1(g_2)H)(g_1^{-1} z) \\
&= (\Sigma_1(g_1)\Sigma_1(g_2)H)(z).
\end{aligned}$$

Therefore, Σ_1 is a representation. To see a unitarity Σ_1, let H_1 and H_2 be elements of $\mathbb{P}_{(1)}$ of degrees 2. Therefore, there exist a unique symmetric tensor $\left(H_{1_{i_1,i_2}}\right)_{1 \le i_1 \le i_2 \le 2}$ such that for all $\bar{z} = (z_1, z_2) \in \mathbb{C}^2$ we have

$$H_1(\bar{z}) = \sum_{i_1,i_2=-1}^{1} H_{1_{i_1,i_2}} z_{i_1} z_{i_2}. \tag{7}$$

In similar way, we also have $H_2(\bar{z}) = \sum_{i_1,i_2=-1}^{1} H_{2_{i_1,i_2}} z_{i_1} z_{i_2}$. Now we are ready to define an inner product on the polynomial space $\mathbb{P}_{(1)}$ in the following form:

$$\langle H_1, H_2 \rangle = \sum_{i_1,i_2=-1}^{1} \overline{H_{1_{i_1,i_2}}} H_{2_{i_1,i_2}}. \tag{8}$$

In particular, we obtain

$$\langle H, H \rangle = \sum_{i=-1}^{1} \overline{H_i} H_i = \frac{1}{2}(|z_1|^2 + |z_2|^2)^2.$$

Then, we have

$$\langle (\Sigma_1(g)H, (\Sigma_1(g)H \rangle = \sum_{i=-1}^{1} |\Sigma_1(g)H_i(z_1,z_2)|^2 = \frac{1}{2}(|\bar{x}_1 z_1 - x_2 z_2|^2 + |\bar{x}_2 z_1 + x_1 z_2|^2)^2$$

$$= \frac{1}{2}(|\bar{x}_1 z_1|^2 + |x_2 z_2|^2 - 2|\bar{x}_1 z_1||x_2 z_2| + |\bar{x}_2 z_1|^2 + |x_1 z_2|^2$$
$$+ 2|\bar{x}_2 z_1||x_1 z_2|)^2 = (|\bar{x}_1 z_1|^2 + |x_2 z_2|^2 + |\bar{x}_2 z_1|^2 + |x_1 z_2|^2)^2$$
$$= \frac{1}{2}((|z_1|^2 + |z_2|^2)(|x_1|^2 + |x_2|^2))^2 = \frac{1}{2}(|z_1|^2 + |z_2|^2)^2 = \langle H, H \rangle.$$

Thus, the representation Σ_1 is unitar. The second, we shall show that Σ_1 is irreducible. Namely, we shall prove that $\mathbb{P}_{(1)}$ has no nontrivial invariant subspaces. In the other words, we shall show that if $0 \neq W \subseteq \mathbb{P}_{(1)}$ is invariant subspace then $W = \mathbb{P}_{(1)}$. But in this case, we shall use Proposition 2. The form $g = \begin{pmatrix} x_1 & x_2 \\ -\bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathrm{SU}(2)$ can be decomposed as follows:

$$g = \begin{pmatrix} x_1 & x_2 \\ -\bar{x}_2 & \bar{x}_1 \end{pmatrix} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}. \tag{9}$$

Let $\psi(\theta) := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ and $\Psi(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$. Then the equation (9) can be re-written as the following form:

$$g = \psi\left(-\frac{\theta}{2}\right) \Psi\left(-\frac{\alpha}{2}\right) \psi\left(-\frac{\gamma}{2}\right). \tag{10}$$

From the previous proof, we have alreade proven that Σ_1 is unitar. Since H is a linear combinations of H_{-1}, H_0, and H_1 then we can compute the representation Σ_1 on each H_{-1}, H_0, and H_1. Namely, we have:

$$(\Sigma_1(g)H_{-1})(z_1,z_2) = \mathrm{H}_{-1}\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \frac{1}{\sqrt{2}} k_0 (\bar{x}_1 z_1 - x_2 z_2)^2$$

$$= H_{-1}(z_1,z_2)S^1_{-1,-1}(g) + H_0(z_1,z_2)S^1_{0,-1}(g) + H_1(z_1,z_2)S^1_{1,-1}(g),$$

$$(\Sigma_1(g)H_0)(z_1,z_2) = \mathrm{H}_0\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = k_1(\bar{x}_1 z_1 - x_2 z_2)(\bar{x}_2 z_1 + x_1 z_2)$$

$$= H_{-1}(z_1,z_2)S^1_{-1,0}(g) + H_0(z_1,z_2)S^1_{0,0}(g) + H_1(z_1,z_2)S^1_{1,0}(g),$$

$$(\Sigma_1(g)H_1)(z_1,z_2) = \mathrm{H}_1\left(g^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}k_2(\,\bar{x}_2 z_1 + x_1 z_2)^2$$

$$= H_0(z_1,z_2)S^1_{0,1}(g) + H_{-1}(z_1,z_2)S^1_{-1,1}(g) + H_1(z_1,z_2)S^1_{1,1}(g). \tag{11}$$

Let $g \coloneqq \psi\left(-\frac{\theta}{2}\right) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$ and by using the equation (10), then we have:

$$S^1_{-1,-1}\left(\psi\left(-\frac{\theta}{2}\right)\right) = e^{-i\theta}, S^1_{0,0}\left(\psi\left(-\frac{\theta}{2}\right)\right) = 1, S^1_{1,1}\left(\psi\left(-\frac{\theta}{2}\right)\right) = e^{i\theta}. \tag{12}$$

Moreover, by equating of elements of matrix $S^1(g)M$ and $MS^1(g)$ then we have the equality of elements of M. Namely, $M_{ii} = M_{kk}$ and $M = zI_2$. Thus, Σ_1 is irreducible.

As mentioned before, IUR of Lie algebra $\mathfrak{su}(2)$ arises from IUR of $\mathrm{SU}(2)$. The result of IUR of $\mathfrak{su}(2)$ realized on the space $\mathbb{P}_{(1)}$ can be found in (Berndt, 2007) p. 71. In our result, we confirm the IUR of matrix Lie algebra $\mathfrak{su}(2)$ which is realized on the space $\mathbb{P}_{(2)}$ (Berndt, 2007) p. 72. Our second result is stated in the following proposition.

Propostion 4. *Let* Σ_2 *be the IUR of* $\mathrm{SU}(2)$ *realized on the space* $\mathbb{P}_{(2)} = \langle J_{-2}, J_{-1}, J_0, J_1, J_2 \rangle$ *where*

$$J_{-2}(z_1,z_2) = \frac{1}{2\sqrt{6}}z_2^4, J_{-1}(z_1,z_2) = \frac{1}{\sqrt{6}}z_1 z_2^3$$

$$J_0(z_1,z_2) = \frac{1}{2}z_1^2 z_2^2, J_1 = \frac{1}{2\sqrt{3}}z_1^3 z_2, J_2 = \frac{1}{2\sqrt{6}}z_1^4.$$

Then the IUR of $\mathfrak{su}(2) = \langle B_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, B_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\rangle$ *on the space* $\mathbb{P}_{(2)}$ *can be written in the following forms*

$$\mathrm{d}\Sigma_2(B_1)J_{-2}) = iJ_{-1}, \mathrm{d}\Sigma_2(B_2)J_{-2}) = -J_{-1}, \mathrm{d}\Sigma_2(B_3)J_{-2}).$$

$$\mathrm{d}\Sigma_2(B_1)J_{-1}) = iJ_{-2} + \frac{3i}{\sqrt{6}}J_0, \mathrm{d}\Sigma_2(B_2)J_{-1}) = J_{-2} - \frac{3}{\sqrt{6}}J_0, \mathrm{d}\Sigma_2(B_3)J_{-1}) = iJ_{-1}.$$

$$\mathrm{d}\Sigma_2(B_1)J_0) = \frac{i\sqrt{6}}{2}J_{-1} + i\sqrt{3}J_1, \mathrm{d}\Sigma_2(B_2)J_0) = \frac{\sqrt{6}}{2}J_{-1} - \sqrt{3}J_1,\ \ \mathrm{d}\Sigma_2(B_3)J_0) = 0.$$

$$\mathrm{d}\Sigma_2(B_1)J_1) = \frac{3i}{2\sqrt{3}}J_0 + \frac{i\sqrt{2}}{2}2, \mathrm{d}\Sigma_2(B_2)J_1) = \frac{3}{2\sqrt{3}}J_0 - \frac{1}{2}J_1, \mathrm{d}\Sigma_2(B_3)J_1) = -iJ_1.$$

$$\mathrm{d}\Sigma_2(B_1)J_2) = \frac{-2i}{\sqrt{2}}J_1, \mathrm{d}\Sigma_2(B_2)J_2) = \frac{2}{\sqrt{2}}J_1, \mathrm{d}\Sigma_2(B_3)J_2) = -2iJ_2.$$

(13)

Proof. Let J be elements of a basis $S = \{J_{-2}, J_{-1}, J_0, J_1, J_2\} \subseteq \mathbb{P}_{(2)}$ where

$$J_{-2}(z_1, z_2) = \frac{1}{2\sqrt{6}} z_2^4, J_{-1}(z_1, z_2) = \frac{1}{\sqrt{6}} z_1 z_2^3$$

$$J_0(z_1, z_2) = \frac{1}{2} z_1^2 z_2^2, J_1 = \frac{1}{2\sqrt{3}} z_1^3 z_2, J_2 = \frac{1}{2\sqrt{6}} z_1^4$$

Let $B = \left\{B_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, B_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\}$ and their 1-parameter subgroups $\tau: \mathbb{R} \ni \theta \mapsto \tau(\theta) = e^{\theta B_i} \in \mathrm{SU}(2), i = 1,2,3$ are given by the following forms (Berndt, 2007)

$$e^{\theta B_1} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, e^{\theta B_2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, e^{\theta B_2}$$
$$= \begin{pmatrix} \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} \end{pmatrix}$$

Then by using equation (3) we have

$$\mathrm{d}\Sigma_2(B_1)J_{-2}) \coloneqq \frac{d}{d\theta}\ \Sigma_2\left(\mathrm{e}^{\theta \mathrm{B}_1}\right)\mathrm{J}_{-2}|_{\theta=0}, \qquad J_{-2} \in \mathrm{S} = \mathfrak{H}^{\infty}$$

where Σ_2 the IUR of $\mathrm{SU}(2)$ on the homogeneous polynomials $\mathbb{P}_{(2)} = \langle J_{-2}, J_{-1}, J_0, J_1, J_2\rangle$. By direct computations then we obtain

$$\Sigma_2\left(\mathrm{e}^{\theta \mathrm{B}_1}\right)J_{-2}(z_1, z_2) = J_{-2}\left(\mathrm{e}^{-\theta \mathrm{B}_1}(z_1, z_2)\right) = J_{-2}\left(\begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) =$$
$$J_{-2}\begin{pmatrix} z_1\cos\frac{\theta}{2} + i\, z_2\sin\frac{\theta}{2} \\ i\, z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2} \end{pmatrix} = \frac{1}{2\sqrt{6}}\left(i\, z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^4.$$

Then the explisit formula for $\mathrm{d}\Sigma_2(B_1)J_{-2}) = \frac{d}{d\theta}\ \Sigma_2\left(\mathrm{e}^{\theta \mathrm{B}_1}\right)\mathrm{J}_{-2}|_{\theta=0}$ is given as follows:

$$\mathrm{d}\Sigma_2(B_1)J_{-2}) = \frac{4}{2\sqrt{6}}\left(i\, z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^4\left(\frac{i}{2}\, z_1\cos\frac{\theta}{2} - \frac{z_2}{2}\sin\frac{\theta}{2}\right)|_{\theta=0}$$
$$= \frac{4}{2\sqrt{6}}(z_2)^4\left(\frac{i}{2}\, z_1\right) = \frac{1}{\sqrt{6}} i z_1 z_2^4 = iJ_{-1}.$$

Similarly we also get the following forms:

$$\Sigma_2\left(e^{\theta B_2}\right)J_{-2})(z_1,z_2) = J_{-2}(e^{-\theta B_2}(z_1,z_2) = J_{-2}\left(\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

$$= J_{-2}\begin{pmatrix} z_1\cos\frac{\theta}{2} + \ z_2\sin\frac{\theta}{2} \\ -\ z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2} \end{pmatrix} = \frac{1}{2\sqrt{6}}\left(-\ z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^4.$$

$$\mathrm{d}\Sigma_2(B_2)J_{-2}) = \frac{4}{2\sqrt{6}}\left(-z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^4\left(\frac{-z_1}{2}\cos\frac{\theta}{2} - \frac{z_2}{2}\sin\frac{\theta}{2}\right)|_{\theta=0}$$

$$= \frac{4}{2\sqrt{6}}(z_2)^4\left(\frac{-z_1}{2}\right) = \frac{-1}{\sqrt{6}}z_1 z_2^4 = -J_{-1}.$$

$$\Sigma_2\left(e^{\theta B_3}\right)J_{-2})(z_1,z_2) = J_{-2}(e^{-\theta B_3}(z_1,z_2) = J_{-2}\left(\begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = J_{-2}\begin{pmatrix} z_1 e^{-\frac{i\theta}{2}} \\ z_2 e^{\frac{i\theta}{2}} \end{pmatrix}$$

$$= \frac{1}{2\sqrt{6}}\left(z_2 e^{\frac{i\theta}{2}}\right)^4 = \frac{1}{2\sqrt{6}}z_2^4 e^{i2\theta}.$$

$$\mathrm{d}\Sigma_2(B_3)J_{-2}) = \frac{1}{\sqrt{6}}z_2^4 e^{i2\theta}|_{\theta=0} = \frac{1}{\sqrt{6}}z_2^4 = 2\frac{1}{2\sqrt{6}}z_2^4 = 2J_{-2}.$$

$$\Sigma_2\left(\mathrm{e}^{\theta\mathrm{B}_1}\right)J_{-1}(z_1,z_2) = J_{-1}\left(\mathrm{e}^{-\theta\mathrm{B}_1}(z_1,z_2)\right) = J_{-1}\left(\begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

$$= J_{-1}\begin{pmatrix} z_1\cos\frac{\theta}{2} + i\ z_2\sin\frac{\theta}{2} \\ i\ z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}}\left(z_1\cos\frac{\theta}{2} + i\ z_2\sin\frac{\theta}{2}\right)\left(i\ z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^3.$$

$$\mathrm{d}\Sigma_2(B_1)J_{-1}) = \frac{1}{\sqrt{6}}\left(\frac{-z_1}{2}\sin\frac{\theta}{2} + \frac{iz_2}{2}\cos\frac{\theta}{2}\right)\left(iz_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^3$$

$$+ \frac{3}{\sqrt{6}}\left(z_1\cos\frac{\theta}{2} + iz_2\sin\frac{\theta}{2}\right)\left(iz_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^2\left(\frac{iz_1}{2}\cos\frac{\theta}{2}\right.$$

$$\left.-\frac{z_2}{2}\sin\frac{\theta}{2}\right)|_{\theta=0} = \frac{1}{\sqrt{6}}\left(\frac{iz_2}{2}\right)(z_2)^3 + \frac{3}{\sqrt{6}}(z_1)(z_2)^2\left(\frac{iz_1}{2}\right)$$

$$= i\frac{1}{2\sqrt{6}}z_2^4 + \frac{3i}{\sqrt{6}}.\frac{1}{2}z_1^2 z_2^2 = iJ_{-2} + \frac{3i}{\sqrt{6}}J_0.$$

$$\Sigma_2\left(e^{\theta B_2}\right)J_{-1})(z_1,z_2) = J_{-1}(e^{-\theta B_2}(z_1,z_2) = J_{-1}\left(\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

$$= J_{-1}\begin{pmatrix} z_1\cos\frac{\theta}{2} + \ z_2\sin\frac{\theta}{2} \\ -\ z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}}\left(z_1\cos\frac{\theta}{2} + \ z_2\sin\frac{\theta}{2}\right)\left(-\ z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^3.$$

$$\mathrm{d}\Sigma_2(B_2)J_{-1}) = \frac{1}{\sqrt{6}}\left(\frac{-z_1}{2}\sin\frac{\theta}{2} + \frac{z_2}{2}\cos\frac{\theta}{2}\right)\left(-z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^3$$

$$+ \frac{3}{\sqrt{6}}\left(z_1\cos\frac{\theta}{2} + z_2\sin\frac{\theta}{2}\right)\left(-z_1\sin\frac{\theta}{2} + z_2\cos\frac{\theta}{2}\right)^2\left(-\frac{z_1}{2}\cos\frac{\theta}{2}\right.$$

$$\left.-\frac{z_2}{2}\sin\frac{\theta}{2}\right)|_{\theta=0} = \frac{1}{\sqrt{6}}\left(\frac{z_2}{2}\right)(z_2)^3 + \frac{3}{\sqrt{6}}(z_1)(z_2)^2\left(-\frac{z_1}{2}\right)$$

$$= \frac{1}{2\sqrt{6}}z_2^4 - \frac{3}{\sqrt{6}}\frac{1}{2}z_1^2z_2^2 = J_{-2} - \frac{3}{\sqrt{6}}J_0.$$

$$\Sigma_2\left(e^{\theta B_3}\right)J_{-1})(z_1,z_2) = J_{-1}(e^{-\theta B_3}(z_1,z_2) = J_{-1}\left(\begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = J_{-1}\begin{pmatrix} z_1e^{-\frac{i\theta}{2}} \\ z_2e^{\frac{i\theta}{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}}\left(z_1e^{-\frac{i\theta}{2}}\right)\left(z_2e^{\frac{i\theta}{2}}\right)^3\frac{1}{\sqrt{6}}z_1z_2^3e^{i\theta}$$

$$\mathrm{d}\Sigma_2(B_3)J_{-1}) = \frac{i}{\sqrt{6}}z_1z_2^3e^{i\theta}|_{\theta=0} = \frac{i}{\sqrt{6}}z_1z_2^3 = iJ_{-1}.$$

We continue these computations in similar way, then we have the results as follows:

$$\mathrm{d}\Sigma_2(B_1)J_0) = \frac{i\sqrt{6}}{2}J_{-1} + i\sqrt{3}J_1, \mathrm{d}\Sigma_2(B_2)J_0) = \frac{\sqrt{6}}{2}J_{-1} - \sqrt{3}J_1, \ \ \mathrm{d}\Sigma_2(B_3)J_0) = 0.$$

$$\mathrm{d}\Sigma_2(B_1)J_1) = \frac{3i}{2\sqrt{3}}J_0 + \frac{i\sqrt{2}}{2}2, \mathrm{d}\Sigma_2(B_2)J_1) = \frac{3}{2\sqrt{3}}J_0 - \frac{1}{2}J_1, \mathrm{d}\Sigma_2(B_3)J_1) = -iJ_1.$$

$$\mathrm{d}\Sigma_2(B_1)J_2) = \frac{-2i}{\sqrt{2}}J_1, \mathrm{d}\Sigma_2(B_2)J_2) = \frac{2}{\sqrt{2}}J_1, \mathrm{d}\Sigma_2(B_3)J_2) = -2iJ_2.$$

CONCLUSION

We proved that the representation of $\mathrm{SU}(2)$ realized on the space $\mathbb{P}_{(1)}$ is irreducible unitary representation (IUR). Moreover, we presented the IUR of $\mathfrak{su}(2) = \langle B_1 =$

$\frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, B_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\rangle$ on the space $\mathbb{P}_{(2)}$ which arises from the IUR of $\mathrm{SU}(2)$ on the space $\mathbb{P}_{(2)}$.

REFERENCES

Bayard, P., Roth, J., & Zavala Jiménez, B. (2017). Spinorial representation of submanifolds in metric Lie groups. *Journal of Geometry and Physics*, *114*, 348–374. https://doi.org/10.1016/j.geomphys.2016.12.011

Behzad, O., Contiero, A., & Martins, D. (2022). On the vertex operator representation of Lie algebras of matrices. *Journal of Algebra*, *597*, 47–74. https://doi.org/10.1016/j.jalgebra.2022.01.009

Berndt, R. (2007). *Representations of Linear Groups*. Vieweg. https://doi.org/10.1007/978-3-8348-9401-4

Braslavsky, I., & Stavans, J. (2018). Application of Algebraic Topology to Homologous Recombination of DNA. *IScience*, *4*, 64–67. https://doi.org/10.1016/j.isci.2018.05.008

Cowling, M. G. (2023). Decay estimates for matrix coefficients of unitary representations of semisimple Lie groups. *Journal of Functional Analysis*, *285*(8), 110061. https://doi.org/10.1016/j.jfa.2023.110061

Eswara Rao, S. (2023). Hamiltonian extended affine Lie algebra and its representation theory. *Journal of Algebra*, *628*, 71–97. https://doi.org/10.1016/j.jalgebra.2023.02.031

Fayazi, F., Gholami, A., & Ashrafi, A. R. (2021). A Lie Algebra Approach on Maximal Self Complementary C 3-Codes. In *MATCH Communications in Mathematical and in Computer Chemistry MATCH Commun. Math. Comput. Chem* (Vol. 85).

Feger, R., Kephart, T. W., & Saskowski, R. J. (2020). LieART 2.0 – A Mathematica application for Lie Algebras and Representation Theory. *Computer Physics Communications*, *257*, 107490. https://doi.org/10.1016/j.cpc.2020.107490

Gudmundsson, S. (2017). Biharmonic functions on the special unitary group SU(2). *Differential Geometry and Its Applications*, *53*, 137–147. https://doi.org/10.1016/j.difgeo.2017.05.011

Hampsey, M., van Goor, P., Banavar, R., & Mahony, R. (2024). Exploiting Equivariance in the Design of Tracking Controllers for Euler-Poincare Systems on Matrix Lie Groups. *IFAC-PapersOnLine*, *58*(6), 333–338. https://doi.org/10.1016/j.ifacol.2024.08.303

Hu, P., Kriz, I., & Somberg, P. (2019). Derived representation theory of Lie algebras and stable homotopy categorification of sl. *Advances in Mathematics*, *341*, 367–439. https://doi.org/10.1016/j.aim.2018.10.044

Hussain, A., Usman, M., Zidan, A. M., Sallah, M., Owyed, S., & Rahimzai, A. A. (2024). Dynamics of invariant solutions of the DNA model using Lie symmetry approach. *Scientific Reports*, *14*(1). https://doi.org/10.1038/s41598-024-59983-8

Liebeck, M. W., Shalev, A., & Tiep, P. H. (2020). Character ratios, representation varieties and random generation of finite groups of Lie type. *Advances in Mathematics*, *374*, 107386. https://doi.org/10.1016/j.aim.2020.107386

Lim, N., & Privault, N. (2016). Analytic bond pricing for short rate dynamics evolving on matrix Lie groups. *Quantitative Finance*, *16*(1), 119–129. https://doi.org/10.1080/14697688.2014.990497

Martínez-Tibaduiza, D., Aragão, A. H., Farina, C., & Zarro, C. A. D. (2020). New BCH-like relations of the su(1,1), su(2) and so(2,1) Lie algebras. *Physics Letters A*, *384*(36), 126937. https://doi.org/10.1016/j.physleta.2020.126937

Nasr-Isfahani, A. R. (2015). On Lie algebras associated with representation-finite algebras. *Journal of Algebra*, *444*, 284–296. https://doi.org/10.1016/j.jalgebra.2015.07.023

Pan, C., Bu, L., Chen, S., Yang, W.-X., Mihalache, D., Grelu, P., & Baronio, F. (2022). General rogue wave solutions under SU(2) transformation in the vector Chen–Lee–Liu nonlinear Schrödinger equation. *Physica D: Nonlinear Phenomena*, *434*, 133204. https://doi.org/10.1016/j.physd.2022.133204

Rabinowitch, A. S. (2024). On a new class of progressive waves in Yang–Mills fields with SU(2) symmetry. *Nuclear Physics B*, *1001*, 116505. https://doi.org/10.1016/j.nuclphysb.2024.116505

Saito, S. (2024). Active SU(2) operation on Poincaré sphere. *Results in Physics*, *59*, 107567. https://doi.org/10.1016/j.rinp.2024.107567

Susanto, N. C. P., Hartati, S. J., & Standsyah, R. E. (2023). Systematic Literature Review: Application of Dynamic Geometry Software To Improve Mathematical Problem-Solving Skills. *Mathline : Jurnal Matematika Dan Pendidikan Matematika*, *8*(3), 857–872. https://doi.org/10.31943/mathline.v8i3.458

Xie, Y., & Zhang, B. (2023). On meridian-traceless SU(2)–representations of link groups. *Advances in Mathematics*, *418*, 108947. https://doi.org/10.1016/j.aim.2023.108947