M A T H L I N E
JURNAL MATEMATIKA DAN PENDIDIKAN MATEMATIKA

ISSN 2502-5872 (Print)
ISSN 2622-3627 (Elektronik)

Volume 9 Number 4, November 2024, 1073-1087

PRODUCT CORDIAL LABELING OF SCALE GRAPH $S_{1,r}(C_3)$ FOR $r \geq 3$

Yanne Irene[1], Gustina Elfiyanti[2*], Winda Ayu Mei Lestari[3], Mahmudi[4], Muhammad Manaqib[5]

[1,2,3,4,5]Departement of Mathematics Faculty of Science and Technology, UIN Syarif Hidayatullah, Jakarta Province, Indonesia

*Correspondence: gustina.elfiyanti@uinjkt.ac.id

ABSTRACT

Graph theory plays a crucial role in various fields, including communication systems, computer networks, and integrated circuit design. One important aspect of this theory is product cordial labeling, which involves assigning labels to the vertices and edges of a graph in a specific way to achieve a balance. Despite extensive research, the product cordial labeling of scale graphs has not been thoroughly explored. This study aims to fill this gap by investigating whether the scale graph $S_{1,r}(C_3)$ can be labeled in a product cordial manner. To achieve this, we followed a three-step methodology: first, we identified the vertices and edge notations of the scale graph $S_{1,r}(C_3)$; second, we assigned binary labels (0 and 1) to each vertex and edge to identify a pattern; and third, we proved that this pattern meets the criteria for product cordial labeling. Our findings reveal that the scale graph $S_{1,r}(C_3)$ does indeed support product cordial labeling, thus confirming it as a product cordial graph. This research not only advances our understanding of graph labeling but also provides practical insights that can be applied to optimize network structures and address complex problems in science and engineering.

Keywords: Product Cordial Labeling, Product Cordial Graph, Scale Graph $S_{1,r}(C_3)$.

How to Cite: Irene, Y., Gustina Elfiyanti, G., Winda Ayu Mei Lestari, W. A. M., Mahmudi, Muhammad Manaqib, M. (2024). Product Cordial Labeling Of Scale Graph $S_{1,r}(C_3)$ For $r \geq 3$. *Mathline: Jurnal Matematika dan Pendidikan Matematika*, *9*(4), 1073-1087. http://doi.org/10.31943/mathline.v9i4.662

PRELIMINARY

In graph theory, researchers explore various topics such as labeling and coloring. An example of using graph coloring for scheduling has been presented (Cipta et al., 2023). Labeling graphs is an essential topic in graph theory. Its focus is on graphs composed of vertices, edges, and a set of numbers called labels. This concept was introduced by Rosa in 1967 (Jesintha et al., 2023). To this day, the application of graph labeling theory plays a significant role, particularly in communication and transportation systems, computer data storage, radar, geographic navigation, and integrated circuit design in electronic components (Agustin et al., 2022).

Graph labeling is a process that maps the set of vertices, edges, or both to the set of natural numbers according to specific rules. It is referred to as vertex labeling when the domain is the set of vertices, edge labeling when the domain is the set of edges, and total labeling when the domain includes both vertices and edges (Ristiawan, 2019). The concept of graph labeling has gained significant popularity in graph theory over the past 60 years due to its broad range of applications (Daisy et al., 2022).

As graph labeling has evolved, product cordial labeling has become a widely studied area. The concept of cordial labeling was first introduced by I. Cahit in 1987 (Palani & Niranjana, 2019). In his paper, Cahit demonstrated that all trees, the complete graph, and the complete bipartite graph exhibit cordial properties (Bosmia & Kanani, 2020). Label each vertex with either a zero or a one. The label of each edge is the absolute difference between the labels of its end vertices (Acharya & Kureethara, 2023).

Let α be a function that maps the vertices of G to $\{0, 1\}$ such that each edge xy is assigned the label $|\alpha(x) - \alpha(y)|$ (Boxwala et al., 2024). The function α is called a cordial labeling of G if the difference between the number of vertices labeled 0 and those labeled 1 is at most 1, and the difference between the number of edges labeled 0 and those labeled 1 is also at most 1 (Daisy et al., 2022). Several new labeling schemes are investigated with minor variations in the concept of cordiality, such as total product cordial labeling, product cordial labeling, and signed product cordial labeling (Sadawarte & Srivastav, 2022). Building on the concept of cordial labeling, Sundaram and his colleagues proposed product cordial labeling. In this variation, the focus shifts from the absolute difference between vertex labels to the product of those labels (Prajapati & Raval, n.d.).

(Gallian, 2022) mentions several graphs have been mentioned that include product cordial labelings, such as, cycles, paths, flower graphs, flower snarks, sunflower graphs, helms, armed helms $W_n \oplus P_2$, closed helms, webs by the duplication of vertices and edges, and lotus inside a circle graph.

A graph $G = (V(G), E(G))$ with a function $\alpha: V(G) \to \{0,1\}$ is termed a binary vertex labeling on G, and $\alpha(v)$ denotes the label of vertex v on G under α. This labeling α induces a labeling on edges $\alpha^*: E(G) \to \{0,1\}$ with $\alpha^*(e = xy) = \alpha(x).\alpha(y)$, known as a product cordial labeling, if $|v_\alpha(0) - v_\alpha(1)| \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| \leq 1$ where $v_\alpha(0)$ represents the number of vertices with label 0, $v_\alpha(1)$ denotes the number of vertices with label 1, $e_\alpha(0)$ indicates the number of edges with label 0, and $e_\alpha(1)$ signifies the number of edges with label 1. A graph with a product cordial labeling is called a product cordial graph (Meena & Usharani, 2022).

This research investigates how graph labeling, specifically product cordial labeling, is applied to scale graphs. Interest in this research is driven by the need to understand how to label graph components efficiently and accurately according to specific rules. Scale graphs, which combine cycle and star graphs, present an interesting case for studying these labeling methods. The phenomena observed include the distribution and balance of labels on graph vertices and edges, as well as adherence to the rules of product cordial labeling.

Many studies of product cordial labeling have been investigated, such as, dragonfly graph by (Harianto, 2018), extensions of burbell graph (Patel et al., 2018), antena graph and tail graph by (Bapat, 2018a), path union of mix graphs (Bapat, 2018b), the double path unions obtained on C_3 related graphs (Bapat, n.d.), double wheel and double fan related graphs (Rokad, 2019), product graphs by (Palani & Niranjana, 2019), the sum and union of two fourth power of paths and cycles graphs (Nada et al., 2019), hypercube related graphs (Gadhiya et al., 2020), jewel graph, jellyfish graph and mongolian tent graph by (Daisy et al., 2022), powers of paths by (Daisy et al., 2022), some graph operations on crown, helm, and wheel graph by (Domingo & Racca, 2022).

Significant progress has been made in studying product cordial labeling in recent years. However, there remains a gap in applying this concept to more complex and diverse graph structures, such as scale graphs. Most recent studies have focused on more straightforward or well-known graphs, leaving the product cordial labeling of scale graphs relatively unexplored. Furthermore, while the theoretical foundations are well established, these labeling techniques' practical implementation and real-world applications have yet to be extensively discussed. This research aims to fill this gap by providing a detailed analysis of product cordial labeling for scale graphs and exploring its practical applications.

This research introduces several new aspects that differentiate it from previous studies. Firstly, it applies product cordial labeling to the scale graph $S_{1,r}(C_3)$, a structure that has not been extensively studied in this context. This approach provides new insights into how product cordial labeling functions in complex graph configurations. Secondly, this study bridges the gap between theoretical concepts and practical applications by demonstrating how these labeling techniques can be implemented in real-world situations. The detailed examination of patterns, conjectures, and their proofs offers a fresh perspective on product cordial labeling, expanding its potential uses and demonstrating its versatility across different graphs.

The scale graph combines of a cycle graph with three nodes and a star graph on the right and left sides of the cycle graph with r nodes. The scale graph always has an odd number of nodes. The scale graph is denoted by $S_{1,r}(C_3)$ (Sari et al., 2013).

METHODS

The research method begins with an in-depth literature review on product cordial labeling, involving previous studies discussing its concepts, definitions, and applications in various graphs. After gaining a comprehensive understanding, we proceeded with the labeling process of the scale graph $S_{1,r}(C_3)$ using labels 0 and 1, where each vertex of graph G is mapped into the set $\{0, 1\}$ so that each edge xy is labeled $\alpha(x)\alpha(y)$.

The next step is to identify the patterns that emerge from this labeling. These patterns are determined by labeling several examples of the scale graph $S_{1,r}(C_3)$. Based on these patterns, we formulate several conjectures that explain the conditions of product cordial labeling on the graph. These conjectures are then tested for validity; if proven true, they will become theorems. However, if proven false, we will formulate new conjectures until we obtain accurate ones.

To solve the research problem, several theorems are required. These theorems include the vertex labeling theorem, which states the conditions for vertex labeling on graph G to be product cordial; the edge labeling theorem, which states the conditions for edge labeling on graph G to be product cordial; and the consistency labeling theorem, which ensures that the difference in the number of vertices and edges labeled 0 and 1 does not exceed 1. This study employs theoretical analysis with a constructive approach, utilizing the basic principles of graph theory and combinatorics to prove the resulting conjectures.
Below is a description of the flowchart for this research project.

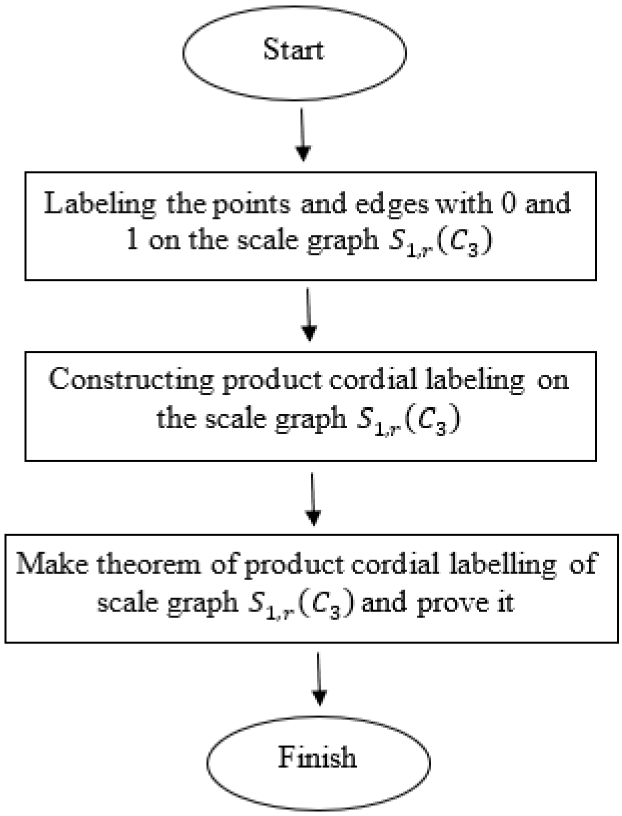

Figure 1. Flowchart

RESULT AND DISCUSSION

The set of vertices of the scale graph $S_{1,r}(C_3)$ is represented by $V\big(S_{1,r}(C_3)\big) = \{x_i|1 \le i \le r\} \cup \{y_i|1 \le i \le r\} \cup \{z_i|0 \le i \le 2\}$ and set of edges denoted by $E\big(S_{1,r}(C_3)\big) = \{x_1x_i|2 \le i \le r\} \cup \{y_1y_i|2 \le i \le r\} \cup \{z_0x_1\} \cup \{z_0y_1\} \cup \{z_0z_i|1 \le i \le 2\} \cup \{z_1z_2\}$. Figure 3 shows the vertices and edges of scale graph $S_{1,r}(C_3)$.

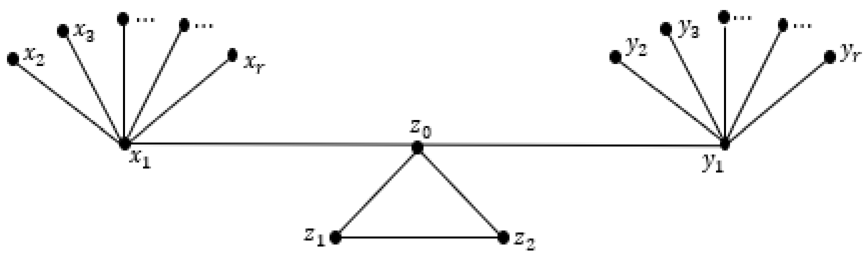

Figure 2. Notational Representation of Vertices and Edges in The Scale Graph $S_{1,r}(C_3)$

To determine the construction of product cordial labeling on the scale graph $S_{1,r}(C_3)$ for $r \geq 3$, we will first establish its labeling pattern through the following lemmas.

Lemma 1 $S_{1,3}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,3}(C_3)$ to prove that the $S_{1,3}(C_3)$ is a product cordial graph.

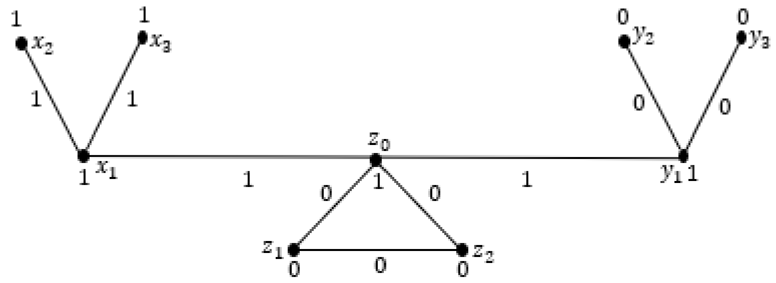

Figure 3. Product Cordial Labeling of Scale Graph $S_{1,3}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq 3$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq 3$$
$$\alpha(z_0z_i) = 0,1 \leq i \leq 2$$
$$\alpha(z_1z_2) = 0$$
$$\alpha(z_0x_1) = 1$$
$$\alpha(z_0y_1) = 1$$
$$\alpha(x_1x_i) = 1,2 \leq i \leq 3$$
$$\alpha(y_1y_i) = 0,2 \leq i \leq 3$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 5 = 3 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 4 = 3 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 4 = 3 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 5 = 3 + 2 = r + 2$.

Thus, $|v_\alpha(0) - v_\alpha(1)| = |4 - 5| = 1 \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| = |5 - 4| = 1 \leq 1$.

The labeling on the scale graph $S_{1,3}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Lemma 2 $S_{1,4}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,4}(C_3)$ to prove that the $S_{1,4}(C_3)$ is a product cordial graph.

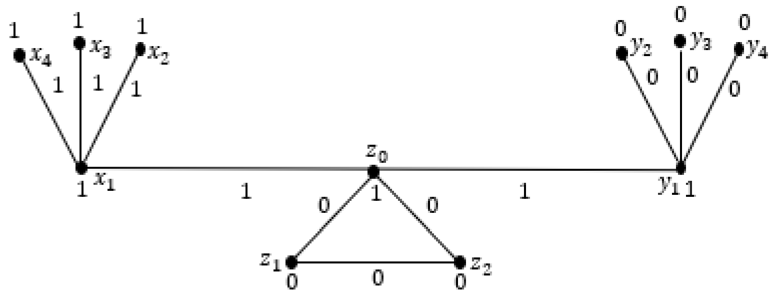

Figure 4. Product Cordial Labeling of Scale Graph $S_{1,4}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq 4$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq 4$$
$$\alpha(z_0 z_i) = 0,1 \leq i \leq 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1,2 \leq i \leq 4$$
$$\alpha(y_1 y_i) = 0,2 \leq i \leq 4$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 6 = 4 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 5 = 4 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 5 = 4 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 6 = 4 + 2 = r + 2$.

Thus, $|v_\alpha(0) - v_\alpha(1)| = |5 - 6| = 1 \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| = |6 - 5| = 1 \leq 1$.

The labeling on the scale graph $S_{1,4}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Lemma 3 $S_{1,7}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,7}(C_3)$ to prove that the $S_{1,7}(C_3)$ is a product cordial graph.

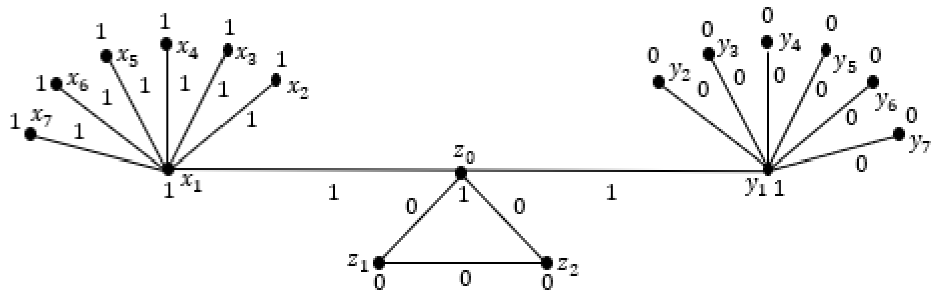

Figure 5. Product Cordial Labeling of Scale Graph $S_{1,7}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq 7$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq 7$$
$$\alpha(z_0 z_i) = 0,1 \leq i \leq 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1,2 \leq i \leq 7$$
$$\alpha(y_1 y_i) = 0,2 \leq i \leq 7$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 9 = 7 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 8 = 7 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 8 = 7 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 9 = 7 + 2 = r + 2$.
Thus, $|v_\alpha(0) - v_\alpha(1)| = |8 - 9| = 1 \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| = |9 - 8| = 1 \leq 1$.
The labeling on the scale graph $S_{1,7}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Lemma 4 $S_{1,8}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,8}(C_3)$ to prove that the $S_{1,8}(C_3)$ is a product cordial graph.

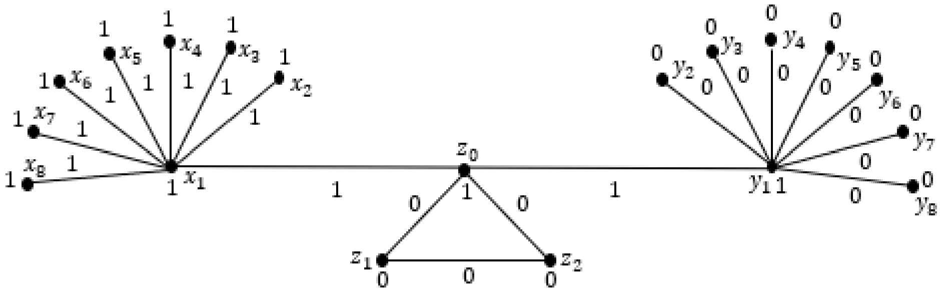

Figure 6. Product Cordial Labeling of Scale Graph $S_{1,8}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq 8$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq 8$$
$$\alpha(z_0 z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1, 2 \leq i \leq 8$$
$$\alpha(y_1 y_i) = 0, 2 \leq i \leq 8$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 10 = 8 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 9 = 8 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 9 = 8 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 10 = 8 + 2 = r + 2$.
Thus, $|v_\alpha(0) - v_\alpha(1)| = |9 - 10| = 1 \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| = |10 - 9| = 1 \leq 1$.
The labeling on the scale graph $S_{1,8}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Lemma 5 $S_{1,11}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,11}(C_3)$ to prove that the $S_{1,11}(C_3)$ is a product cordial graph.

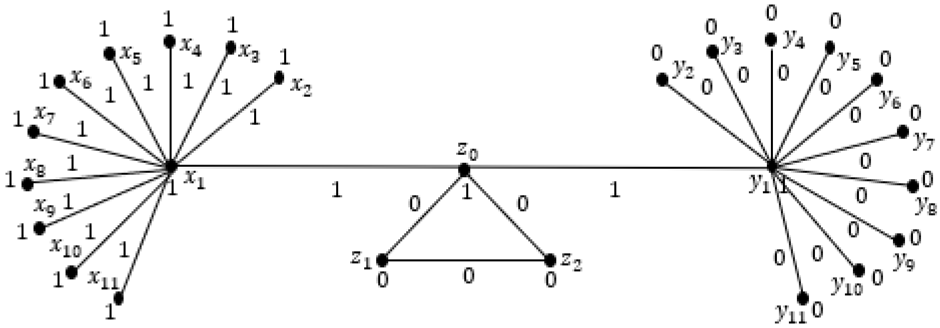

Figure 7. Product Cordial Labeling of Scale Graph $S_{1,11}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq 11$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq 11$$
$$\alpha(z_0 z_i) = 0,1 \leq i \leq 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1,2 \leq i \leq 11$$
$$\alpha(y_1 y_i) = 0,2 \leq i \leq 11$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 13 = 11 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 12 = 11 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 12 = 11 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 13 = 11 + 2 = r + 2$.

Thus, $|v_\alpha(0) - v_\alpha(1)| = |12 - 13| = 1 \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| = |13 - 12| = 1 \leq 1$.

The labeling on the scale graph $S_{1,11}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Lemma 6 $S_{1,12}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,12}(C_3)$ to prove that the $S_{1,12}(C_3)$ is a product cordial graph.

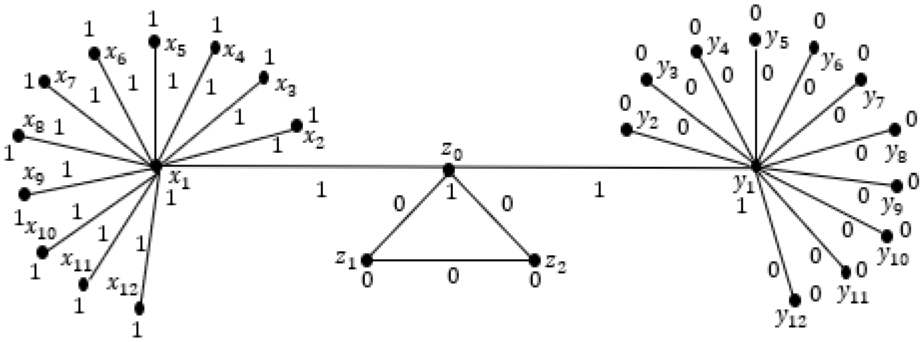

Figure 8. Product Cordial Labeling of Scale Graph $S_{1,12}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \le i \le 2$$
$$\alpha(x_i) = 1, 1 \le i \le 12$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \le i \le 12$$
$$\alpha(z_0 z_i) = 0,1 \le i \le 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1,2 \le i \le 12$$
$$\alpha(y_1 y_i) = 0,2 \le i \le 12$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 14 = 12 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 13 = 12 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 13 = 12 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 14 = 12 + 2 = r + 2$.

Thus, $|v_\alpha(0) - v_\alpha(1)| = |13 - 14| = 1 \le 1$ and $|e_\alpha(0) - e_\alpha(1)| = |14 - 13| = 1 \le 1$. The labeling on the scale graph $S_{1,12}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Lemma 7 $S_{1,15}(C_3)$ is a product cordial graph.

Proof. First, we need to determine the labels 0 and 1 on the $S_{1,15}(C_3)$ to prove that the $S_{1,15}(C_3)$ is a product cordial graph.

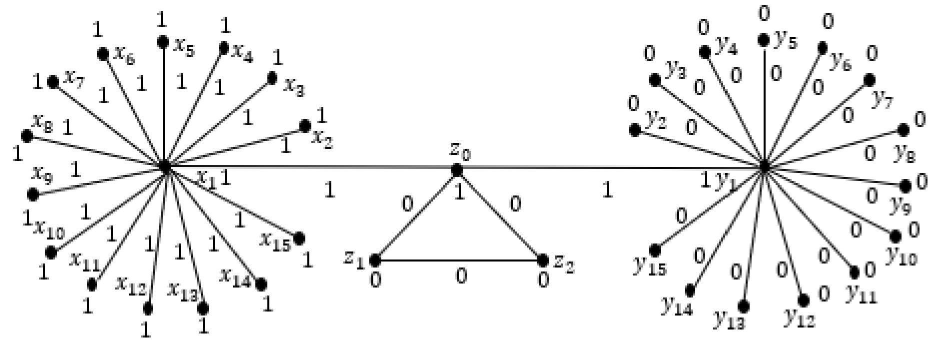

Figure 9. Product Cordial Labeling of Scale Graph $S_{1,15}(C_3)$

Based on the above figure, the graph labeling obtained is as follows:

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq 15$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq 15$$
$$\alpha(z_0 z_i) = 0{,}1 \leq i \leq 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1{,}2 \leq i \leq 15$$
$$\alpha(y_1 y_i) = 0{,}2 \leq i \leq 15$$

With the count of vertices labeled as 1 is $v_\alpha(1) = 17 = 15 + 2 = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = 16 = 15 + 1 = r + 1$, the count of edges labeled as 1 is $e_\alpha(1) = 16 = 15 + 1 = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = 17 = 15 + 2 = r + 2$.

Thus, $|v_\alpha(0) - v_\alpha(1)| = |16 - 17| = 1 \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| = |17 - 16| = 1 \leq 1$. The labeling on the scale graph $S_{1,15}(C_3)$ satisfies the product cordial labeling rules, so it is a product cordial graph.

Based on lemma 1 to 7, the construction of product cordial labeling on the scale graph $S_{1,r}(C_3)$ for $r \geq 3$ will be explained with the following theorem.

Theorem Scale Graph $S_{1,r}(C_3)$ for $r \geq 3$ is a product cordial graph

Proof:

Define $\alpha: V(S_{1,r}(C_3)) \rightarrow \{0,1\}$ as

$$\alpha(z_0) = 1$$
$$\alpha(z_i) = 0, 1 \leq i \leq 2$$
$$\alpha(x_i) = 1, 1 \leq i \leq r$$
$$\alpha(y_i) = 1, i = 1$$
$$\alpha(y_i) = 0, 2 \leq i \leq r$$
$$\alpha(z_0 z_i) = 0,1 \leq i \leq 2$$
$$\alpha(z_1 z_2) = 0$$
$$\alpha(z_0 x_1) = 1$$
$$\alpha(z_0 y_1) = 1$$
$$\alpha(x_1 x_i) = 1,2 \leq i \leq r$$
$$\alpha(y_1 y_i) = 0,2 \leq i \leq r$$

With the labeling α above, we obtain the count of vertices labeled as 1 is $v_\alpha(1) = r + 2$, the count of vertices labeled as 0 is $v_\alpha(0) = r + 1$ the count of edges labeled as 1 is $e_\alpha(1) = r + 1$, and the count of edges labeled as 0 is $e_\alpha(0) = r + 2$. In product cordial labeling, edge labeling results from multiplying the two vertices connected by the edge.

Thus, for this case, it is obtained that $|v_\alpha(0) - v_\alpha(1)| \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| \leq 1$. Therefore, it can be concluded that the scale graph $S_{1,r}(C_3)$ for $r \geq 3$ is a product cordial graph.

CONCLUSION

Based on the previous discussion, we have obtained theorem stating that the scale graph $S_{1,r}(C_3)$ is a product cordial graph because it incorporates the idea of product cordial labeling, which is demonstrated by:

$$\alpha: V(S_{1,r}(C_3)) \rightarrow \{0,1\}$$

with $|v_\alpha(0) - v_\alpha(1)| \leq 1$ and $|e_\alpha(0) - e_\alpha(1)| \leq 1$.

So, we have proven that the scale graph $S_{1,r}(C_3)$ is a product cordial graph.

This study opens up several opportunities for future research. One potential direction is to explore the product cordial labeling of other types of graphs or more complex graph structures. Additionally, investigating the application of product cordial labeling in real-world networks, such as communication or social networks, could provide valuable insights.

However, there are certain limitations to this study. The research primarily focuses on the theoretical aspects of product cordial labeling and its application to the specific case of the scale graph $S_{1,r}(C_3)$. The generalizability of the findings to other graph types or larger graphs has not been extensively explored. Moreover, the practical implementation and computational complexity of achieving product cordial labeling in larger or more intricate graphs were not addressed in this study. Future work should aim to overcome these limitations by expanding the scope of graphs studied and addressing the practical aspects of implementing product cordial labeling.

REFERENCES

Acharya, M., & Kureethara, J. V. (2023). Characterization of product cordial dragon graphs. *Communications in Combinatorics and Optimization*, *9*(4), 805–811. https://doi/10.22049/CCO.2023.27882.1384

Agustin, I. H., Susilowati, L., Dafik, Cangul, I. N., & Mohanapriya, N. (2022). On the vertex irregular reflexive labeling of several regular and regular-like graphs. *Journal of Discrete Mathematical Sciences and Cryptography*, *25*(5), 1457–1473. https://doi.org/10.1080/09720529.2022.2063543

Bapat, M. (2018a). Extended vertex product cordinal (evpc) lebeling of graphs, 3(1), 272–274. https://www.mathsjournal.com/pdf/2018/vol3issue1/PartD/2-6-27-690.pdf

Bapat, M. V. (n.d.). Product Cordial Labeling Of Double Path Union Of C3 Related Graphs, *International Journal of Engineering Technologies and Management Research, 5*(4), 146–152. https://doi/10.5281/zenodo.1251967

Bapat, M. V. (2018b). *Product cordiality of path union of mix graphs*.

Bosmia, M. I., & Kanani, K. K. (2020). Product Cordial Labeling for the Line Graph of Bistar. In *Recent Advancements in Graph Theory* (pp. 75–82). CRC Press.

Boxwala, S., Shinde, P. N., Mundlik, N. D., & Phadke, A. S. (2024). Group A-cordial labeling of some spiders. *Palestine Journal of Mathematics*, *13*(1). https://pjm.ppu.edu/sites/default/files/papers/PJM_13%281%29_2024_25_to_34.pdf

Cipta, H., Widyasari, R., & Batubara, F. H. (2023). Graph Coloring Implementation Using Welch Powell Algorithm In Lecture Scheduling Design For Mathmatics Department. *Mathline: Jurnal Matematika Dan Pendidikan Matematika*, *8*(4), 1383–1398. https://doi.org/10.31943/mathline.v8i4.537

Daisy, K, J., Revina Belasia, W., & Jesmalar, L. (2022). *Product Cordial Labeling of some Graphs*. *13*(2). https://research.holycrossngl.edu.in/Content/Cross_res_2022/dec%202022-117.pdf

Daisy, K. J., Sabibha, R. S., Jeyanthi, P., & Youssef, M. Z. (2022a). k-Product cordial labeling of cone graphs. *International Journal of Mathematical Combinatorics*, *2*, 72–80. https://fs.unm.edu/IJMC/IJMC-2-2022.pdf#page=77

Daisy, K. J., Sabibha, R. S., Jeyanthi, P., & Youssef, M. Z. (2022b). k-Product cordial labeling of powers of paths. *Jordan Journal of Mathematics and Statistics*, *15*(4A), 911–924. https://doi.org/10.47013/15.4.8

Domingo, M., & Racca, A. (2022). Product Cordial Graph in the Context of Some Graph Operations on Crown, Helm and Wheel Graph. *International Mathematical Forum, 17*(2), 89–104. https://doi.org/10.12988/imf.2022.912316

Gadhiya, J. A., Khunti, S. S., & Rupani, M. P. (2020). Product cordial labeling of hypercube related graphs. *Malaya Journal of Matematik*, *8*(04), 1991–1994. https://doi/10.26637/MJM0804/0112

Gallian, J. A. (2022). A Dynamic Survey of Graph Labeling. In *the electronic journal of combinatorics*, 25, 1–623. https://doi.org/10.37236/11668

Harianto, B. (2018). Pelabelan Product Cordial dan Product Cordial Sisi Pada Graf Dragonfly. *LOGIK@*, *8*(1), 11–18. http://repository.uinbanten.ac.id/id/eprint/6005

Jesintha, J. J., Devakirubanithi, D., & Charishma, R. (2023). New types of graphs on cordial labeling. *Bull. Pure Appl. Sci., Sect. E Math. Stat. E*, *42*, 36–42. https://doi/10.48165/bpas.2023.42E.1.5

Meena, S., & Usharani, S. (2022). Product Cordial Labelling for Some Bicyclic Graphs. *Communications in Mathematics and Applications*, *13*(4), 1329. https://doi/10.26713/cma.v13i4.2171

Nada, S., Elrayes, A., Elrokh, A., & Rabie, A. (2019). Signed Product Cordial of the Sum and Union of Two Fourth Power of Paths and Cycles. *Machine Learning Research*, *4*, 45–50. https://doi.org/10.11648/j.mlr.20190404.11

Palani, K., & Niranjana, A. (2019). Product Cordial Labeling Of Product Graphs. *International Journal of Recent Research Aspects ISSN*, 2349–7688. https://www.ijrra.net/Vol6issue1/IJRRA-06-01-01.pdf

Patel, S. K., Prajapati, U. M., & Kansagara, A. N. (2018). Product cordial labeling of extensions of barbell graph. *Mathematical Combinatorics*, *4*, 146–159.

Prajapati, U. M., & Raval, K. K. (n.d.). *Product Cordial Labeling in Context of Some One Point Union Graphs*, *5*(4), 490–495. https://doi.org/10.4236/ojdm.2016.64022

Ristiawan, R. (2019). Graf Pendulum dengan Pelabelan Graceful Super Fibonacci. *Faktor Exacta*, *12*(1), 51–57. https://doi.org/10.30998/faktorexacta.v12i1.3210

Rokad, A. H. (2019). Product cordial labeling of double wheel and double fan related graphs. *Kragujevac Journal of Mathematics*, *43*(1), 7–13. https://www.researchgate.net/publication/332712197_Product_cordial_labeling_of_Double_wheel_and_double_fan_related_graphs

Sadawarte, S., & Srivastav, S. (2022). Signed product cordial labeling of some pan graphs. *Materials Today: Proceedings*, *57*, 2307–2310. https://doi.org/10.1016/j.matpr.2022.01.097

Sari, I., Nugraha, N., & Kosasih, R. (2013). Pelabelan Harmonis pada Graf Timbangan. *UG Journal*, *7*(5), 21–22. https://ejournal.gunadarma.ac.id/index.php/ugjournal/article/view/858